COSETS AND LAGRANGE'S THEOREM

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Left Coset:

Let H be a subgroup of a group G. Let $a \in G$. Then the set $aH=\{ah/h \in H\}$ is called the left Coset of H defined by a in G.

Right Coset:

Let H be a subgroup of a group G. Let $a \in G$. Then the set $Ha=\{ha/h \in H\}$ is called the right Coset of H defined by a in G.

Example:

Consider $(Z_{12},\oplus)$. Then H = {0,4,8} is a subgroup of G.The left cosets of H are

0+H = {0,4,8} = H

1+H = {1,5,9}

2+H = {2,6,10}

3+H = {3,7,11}

4+H = {4,8,0}

5+H = {5,9,1}

We notice that 4+H = {4,8,0}=H and

5+H = {5,9,1} = 1+H etc

Theorem:

Let H be a subgroups of G. Then

(i)any two left cosets of H are either identical or disjoint

(ii)Union of all the left cosets of H is G

(iii)the number of elements in any left coset aH is the same as the number of elements in H

Proof of the theorem:

(i)Let aH and bH be two left cosets.Suppose aH and bH are not disjoint.to prove that aH = bH

Since aH and bH are not disjoint $aH \cap bH \neq \emptyset$

$\therefore$ there exists an element $c \in aH \cap bH$

$\therefore$ $c \in aH \cap bH$ and $c \in aH \cap bH$

$\therefore aH = cH$ and $bH = cH$

(by $a \in bH \Leftrightarrow aH = bH$)

$\therefore aH = bH$

(ii)let $a \in G$. Then $a = ae \in aH$

$\therefore$ Every element of G belongs to a left cosets of H.

$\therefore$ The union of all the left cosets of H in G.

(iii)The map $f: H \rightarrow aH$ defined by $f(h) = ah$ is clearly bijection.

Hence every left coset has the same number of elements as H.

Theorem:

Let H be a subgroup of G. the number of left cosets of H is the same as the number of right cosets of H

Proof:

Let L and R be denote the set of left and right cosets of H. we define a map $f:L\rightarrow R$ by

$f(aH) = Ha^{-1}$

$aH = bH \Rightarrow a^{-1}b \in H$

$\Rightarrow a^{-1} \in Hb^{-1}$

$\Rightarrow Ha^{-1} = Hb^{-1}$

$\therefore$ f is well defined.

$f(aH) = f(bH) \Rightarrow Ha^{-1} = Hb^{-1}$

$\Rightarrow a^{-1} \in Hb^{-1}$

$\Rightarrow a^{-1} = Hb^{-1}$

$\Rightarrow a = bh^{-1}$

$\Rightarrow a \in bH$

$\Rightarrow aH = Hb$

$\therefore f$ is 1-1

For, every right coset Ha has a pre image $a^{-1}H$.

∴ f is onto

Hence f is bijection from L to R.

Hence the number of left cosets is same as the number of right cosets

Let H be a subgroup of G,
and a,b in G.

1. aH = bH iff a belongs to bH
2. aH and bH are either equal or disjoint
3. |aH| = |bH|

LAGRANGE'S THEOREM

If G is finite group and H is a subgroup of G, then

a) $|H|$ divides $|G|$.
b) The number of distinct left (right) cosets of H in G is $|G|/|H|$

Proof:

Let $H \leq G$ with $|G| = n$, and $|H| = k$.

Write the elements of H in row 1:

e	h_2	h_3	…	h_k

Choose any a_2 in G not in row 1.

Write a_2H in the second row.

e	h_2	h_3	…	h_k
a_2	$a_2 h_2$	$a_2 h_3$	…	$a_2 h_k$

Continue in a similar manner…

Since G is finite, this process will end

e	h_2	h_3	…	h_k
a_2	$a_2 h_2$	$a_2 h_3$	…	$a_2 h_k$
a_3	$a_3 h_2$	$a_3 h_3$	…	$a_3 h_k$
:	:	:		:
a_r	$a_r h_2$	$a_r h_3$		$a_r h_k$

Rows are disjoint by (2)

Each row has k elements by (3)

e	h_2	h_3	…	h_k
a_2	$a_2 h_2$	$a_2 h_3$	…	$a_2 h_k$
a_3	$a_3 h_2$	$a_3 h_3$	…	$a_3 h_k$
:	:	:		:
a_r	$a_r h_2$	$a_r h_3$		$a_r h_k$

Let r be the number of distinct cosets.

Clearly |G| = |H| • r, and

r = |G|/|H|.

e	h_2	h_3	…	h_k
a_2	$a_2 h_2$	$a_2 h_3$	…	$a_2 h_k$
a_3	$a_3 h_2$	$a_3 h_3$	…	$a_3 h_k$
:	:	:		:
a_r	$a_r h_2$	$a_r h_3$		$a_r h_k$

THANK YOU