
Welcome

METRIC SPACES

Definition:

A **metric space** is a non – empty set M together with a function $d: M \times M \to R$ satisfying the following conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in M$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$
(**triangle inequality**)

d is called a **metric** or **distance function** and $d(x, y)$ is called the **distance** between x and y.

Example: 1

In R we define $d(x, y) = |x - y|$. Then d is a metric on R.

Proof:

(i) Clearly $d(x, y) = |x - y| \geq 0$ for all $x, y \in R$

(ii) $d(x, y) = 0 \Leftrightarrow |x - y| = 0$

$\Leftrightarrow x = y$

(iii) $d(x, y) = |x - y| = |-(y - x)| = |y - x| = d(y, x)$

$\therefore d(x, y) = d(y, x)$

(iv) Let $x, y, z \in R$

$$d(x, z) = |x - z| = |x - y + y - z| \le |x - y| + |y - z|$$

$$\left[\text{Q}\ |x + y| \le |x| + |y| \right]$$

$$= d(x, y) + d(y, z)$$

$\therefore d(x, z) \le d(x, y) + d(y, z)$

Hence d is a metric on R.

This is called the **usual metric** on R.

Example: 2

On any non – empty set M we define d as follows

$$d(x,y)=\begin{cases}0 \text{ if } x=y\\ 1 \text{ if } x\neq y\end{cases}$$

Then d is a metric on M.

Proof:

(i) Clearly $d(x,y)\geq 0$ for all $x,y\in M$

(ii) $d(x,y)=0 \text{ iff } x=y$

(iii) $d(x,y) = d(y,x) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$

$\therefore d(x,y) = d(y,x)$ for all $x, y \in M$

(iv) Let $x, y, z \in M$

Case (i) $x = z$

Then $d(x,z) = 0$

Also, $d(x,y) + d(y,z) \geq 0$

$\therefore d(x,z) \leq d(x,y) + d(y,z)$

Case (ii) $x \neq z$

Then $d(x,z)=1$

Also, since x, z are distinct, y cannot be equal to both x and $z.$

Hence either $y \neq x$ or $y \neq z.$

$\therefore d(x,y)+d(y,z) \geq 1$

$\therefore d(x,z) \leq d(x,y)+d(y,z)$

Thus $d(x,z) \leq d(x,y)+d(y,z)$ for all $x,y,z \in M.$

Hence d is a metric on $M.$

This is called the **discrete metric** on $M.$


Thank You