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Multiple Handed Hexagonal Tile Assembly Model on Interactive Systems

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***Abstract* – Self assembly of DNA structures may be mapped naturally onto the languages of Chomsky hierarchy. This model is an extension of the theory of tiling by Wang tiles to include a specific mechanism for growth based on the physics of molecular self assembly. Recently, Abstract Tile Assembly and Multiple handed Tile Assembly Models have been discussed. Motivated by this study, Hexagonal tile assembly model on finite interactive system has been proposed and the Sierpinski hexagonal patterns are generated through $6-HTAS$**

***Keywords*: Self-assembly, Abstract tile assembly, Multiple handed tile assembly, Interactive system, Hexagonal scenario, Hexagonal tile assembly**

I INTRODUCTION

Tiling has been used for the study of two-dimensional languages [2]. A brief introduction to interactive systems, an abstract mathematical model of agents' behavior and their interaction is proposed and a research programme for getting structural characterizations for two-dimensional languages generated by self-assembling tiles is projected in [5]. Motivated by this study, Self Assembling Hexagonal Tile System is defined and the languages are recognized by the system of recursive equations using variables representing sets of two dimensional hexagonal pictures and regular expressions. Hexagonal finite interactive system scenario for the new type of hexagonal regular expressions and patterns has been proposed [3]. In [4], hexagonal tiles and scenarios have been introduced based on tiling hexagonal unit cells with colors representing two dimensional hexagonal pictures. A complete 3×3 Hexagonal Finite Interactive System ($HFIS$) has been defined and the two dimensional hexagonal languages are recognized by the corresponding regular expressions.

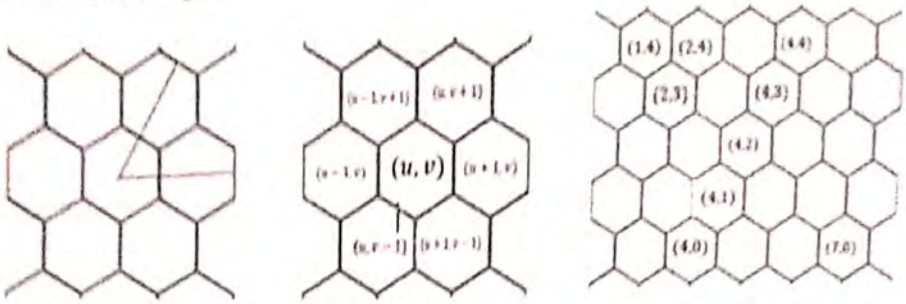
On the other hand, The tile assembly of the discrete Sierpinski triangle by squared tiles called the h - handed assembly model ($h - HAM$) is provided in [1]. They have analyzed and exposed the incapability of strictly assembling Sierpinski triangle by $aTAM$ and there is a need for multiple hands. [1].

In this paper, Hexagonal tile assembly model on finite interactive system has been proposed and the Sierpinski hexagonal patterns are generated through $6-HTAS$.

II PRELIMINARIES

A. Hexagonal Grid [6]

Hexagonal Grid is an alternative representation of pixel tessellation scheme for the conventional square grid for sampling and representing discredited images. Each pixel is represented by a horizontal deflection followed by a deflection upward and to the right. These directions are represented by a pair of unit vectors u and v. We refer to this coordinate system as the "h_2" system. Given a pixel with coordinates u, v (assumed integer), the coordinates of the neighbors are illustrated in Fig. 1.



(a) (b) (c)

Fig. 1 a) A coordinate system based on unit vectors u and v, (b) the neighborhood of a hexagonal pixel and (c) labeling of a hexagonal pixel

B. Hexagonal Tiles and Scenarios [4]

Let Σ be a finite alphabet. A hexagonal tile is a hexagonal cell labelled with symbol from the given alphabet and enriched with additional information on each border. This information is represented abstractly as an element from a finite set and is called a border label. The role of border labels is to impose local glueing constraints on self-assembling tiles: two neighbouring cells, sharing a side border (east-west or north east -south west or north west - south east) should agree on the label on that border.

Hexagonal scenario is similar to a two-dimensional hexagonal picture, but: (1) each hexagonal cell is replaced by a tile and (2) east-west or north east -south west or north west - south east neighbouring cells have the same label on the common border.

Graphically, a hexagonal scenario is obtained using the hexagonal tiles representing the transitions and identifying the matching classes or states of the neighbouring cells. The labels on the north east and north west borders represent north memory states, while the south east and south west borders represent south memory states and the ones on the west and east borders represent interaction classes. The selected labels on the external borders are called initial for south west, west and north west borders and final for north east, east and south east borders.

To construct the hexagonal assembly by assembling hexagonal tiles, we make use of the following. N is the set of natural numbers $\{0, 1, 2 \ldots\}$, $Z = N \cup -N$ is the set of integers and R is the set of real numbers. We will be working in the two-dimensional hexagonal grid of integer positions $Z \times Z$.

The directions $\mathcal{D} = \{EE, WW, NE, NW, SE, SW\}$ will be used as functions from $Z \times Z$ to $Z \times Z$. A point on the side borders in a unit cell is specified by its middle points such that $EE(x,y) = (x+1, y)$, $WW(x,y) = (x-1, y)$, $NE(x,y) = (x, y+1)$, $NW(x,y) = (x-1, y+1)$, $SE(x,y) = (x+1, y-1)$ an $SW(x,y) = (x, y-1)$. We say that (x,y) and (x',y') are neighbors if $(x',y') \in \{EE(x,y)/WW(x,y)/NE(x,y)/NW(x,y)/SE(x,y)/SW(x,y)\}$. Note that $EE = WW^{-1}$, $NE = SW^{-1}$ and $NW = SE^{-1}$. Examples of tiles and scenarios are presented in Fig. 2.

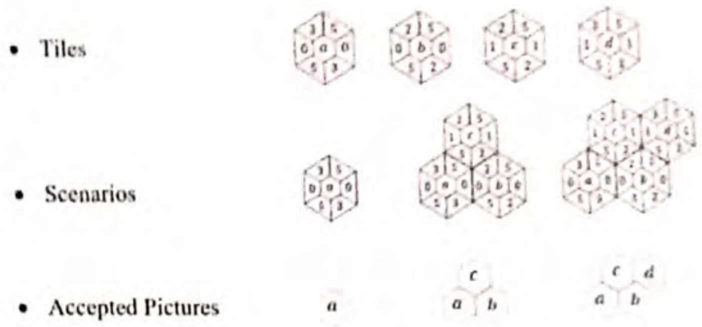


Fig 2. Tiles and Scenarios

C. Interactive System on a Hexagonal Grid

Let Σ be a finite alphabet. An interactive system on a hexagonal grid in a two dimensional plane $HIS = (\Sigma, C, S, R, I, F)$ is a finite hyper-graph with three types of vertices (classes (C) and states (S)) denoting class/state transformation and one type of edges. The first type of vertices is labelled by lowercase letter alphabet; the second type of vertices by upper case letter alphabet and the third type by numbers. The actions/transitions are labeled by small letters in the hexagonal unit cell binding to the transformation rules (R). Each transition has three incoming arrows (edges): one from the class vertex and two from the state vertices. Similarly each transition has three outgoing arrows: one to the class vertex and two to the state vertices. The initial class and states (I) are represented by small arrows. The final class and states (F) are represented by double circles.

D. Example

An interactive system for parallelogram arrays $HIS = (\Sigma, C, S, R, I, F)$ is shown in Fig. 3 (left) where $\Sigma = \{a, a, b, A, B, 1, 2\}$; $C = \{1, 2\}$; $S = \{a, b, A, B\}$; $I = \{a, 1, A\}$; $F = \{b, 2, B\}$; $R = \{X : (a|1|A) \rightarrow (b|2|A)$; $Y : (b|2|A) \rightarrow (a|1|A)$; $Z : (a|2|A) \rightarrow (a|2|A)\}$. This HIS recognizes the language $L_p(HIS)$ consisting of arrays of the form as in Fig. 3. (right).

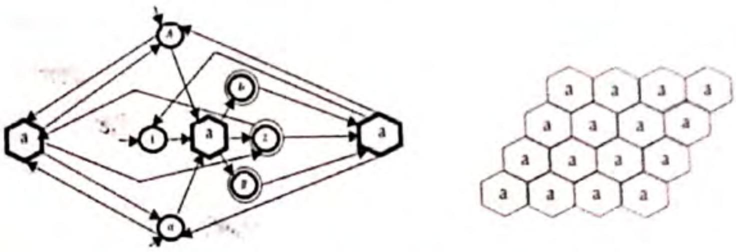


Fig. 3 : An Interactive System for Parallelogram Arrays

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III HEXAGONAL TILE ASSEMBLY MODEL

A. *Strength Function*

A binding domain determine the interaction between tiles when two tiles are placed to each other. A function s : $\Gamma \times \Gamma \rightarrow R$ where $null \in \Gamma$, is a strength function if for all $g, g' \in \Gamma, s(g, g') = s(g', g)$ and $s(null, g) = 0$. Two tiles that abut on sides labelled g and g' bind with strength $s(g, g')$. We will consider s such that mismatched sides have no interaction strength and matching sides have positive strength in which the strength of the side labelled g is $\hat{s}(g) \in N$ and

$$s(g, g') = \begin{cases} \hat{s}(g) & if \quad g = g' \\ 0 & otherwise \end{cases}$$

Let H be a set of hexagonal tiles containing the special tile empty. A configuration of H is a function $A : Z \times Z \rightarrow H$. We write $(x, y) \in A$ iff $A(x, y) \neq$ empty. For $D \in \mathcal{D}$, we write $bd_D(t)$ to refer the binding of the respective side of tile t. We say that the tiles at (x, y) and $D(x, y)$ bind to each other with strength $s_D^A(x, y) = s(bd_D(A(x, y)), bd_{D^{-1}}(A(D(x, y))))$. Thick sides have strength 2, thin sides have strength. If $s_D^A(x, y) > 0$ then the tiles make a bond. $A_t^{(x,y)}$ is the configuration such that $A_t^{(x,y)}(x, y) = t$ and all other sites are empty. $A_{empty}^{(0,0)}$ is called the empty configuration.

B. *Temperature*

A temperature τ gives the minimal interaction strength required to overcome thermal disruption. A configuration C is a stable assembly if for all non empty configurations A and B such that $C = A + B$, $G(C) \geq G(A) + G(B) + \tau$. For $\tau > 0$, a $\tau-$stable assembly must contain a single connected component. For a given assembly, define the bond graph G to be the weighted graph γ in which each element of γ is a vertex, and each edge weight between tiles is the sum of the strengths of the overlapping, matching glue points of the two tiles. An assembly C is said to be $\tau-$stable for positive integer τ if the bond graph G_C has min-cut at least τ and τ unstable otherwise. If the set of border points of all tiles in an assembly is not a connected set then the assembly cannot be $\tau-$stable (for positive). Only overlapping glues that are the same type contribute a non-zero weight, whereas overlapping, non-equal glues always contribute zero weight to the bond graph.

C. *Hexagonal Tile Assembly System*

Let Σ be a finite alphabet. Hexagonal Tile Assembly System $(HTAS)$ on a hexagonal grid is a construct $\Pi = (\Sigma, \Gamma, H, S, I, F, \tau)$ where H is a finite set of Hexagonal tiles over Σ representing the transitions from one state/class to another. $S \in H$ is the seed assembly. I represents the initial states and initial class whereas F represents the final states and final class. τ is a positive integer called the temperature. Γ is a finite set of binding domains of tiles $t \in H$ representing the glue type assigning the border points with at most one glue type g per edge, with each glue being labelled exactly in the center of the corresponding side border. Each glue type $g \in \Gamma$ has some non-negative integer strength $\hat{s}(g)$.

Formally Γ is a 6 – tuple $(g_{SW}, g_{WW}, g_{NW}, g_{NE}, g_{EE}, g_{SE}) \in \Gamma^6$ indicating the binding domains on the South West, West, North West, North East, East and South East sides. According to this definition, tiles may not be rotated; $(g_{SE}, g_{WW}, g_{NW}, g_{NE}, g_{EE}, g_{SW}) \neq (g_{SW}, g_{WW}, g_{NW}, g_{NE}, g_{EE}, g_{SE})$. A special binding domain $null$ represents a non-interaction and the special tile $empty = (null, null, null, null, null, null)$ is used to represent the absence of any other tile.

Hexagonal tile assembly is defined by a relation between configurations. $A \to_{\pi} B$ if there exists a tile $t \in H$ and a site (x, y) such that $B = A + A_t^{(x,y)}$ and B is τ stable. i.e., a hexagonal tile may be added to an assembly if the summed strength of its interactions with its neighbors exceeds a threshold set by the temperature. In particular, at $\tau = 1$, a hexagonal tile may be added if it makes any bond to a neighbour, whereas at $\tau = 2$, the diamond tile to be added must either make two weak bonds or a single strong bond. $\to_T^{*}$ is the reflexive and transitive closure of $\to_T$

Assembly in the $HTAS$ proceeds by growing from the seed assembly S by any sequence of single tile attachments from H so long as each tile attachment connects with strength at least τ. The set of all tiles that are translation of some tile in H are Σ_H^{**}.

D Produced and Terminal hexagonal tile assembly

For a given $\Pi = (\Sigma, \Gamma, H, S, I, F, \tau)$, a partially ordered set the produced hexagonal tile assemblies $Prod_{\Gamma}$ is defined as $Prod_{\Gamma} = \{A \exists S \in H, S s.t. S \to_{\pi}^{*} A\}$ and $A \leq B$ iff $A \to_{\pi}^{*} B$.

We say that $A \to_{\pi}^{1} B$ if A may grow into B through a single tile attachment and we say $A \to_{\pi} B$ if A can grow into B through 0 or more tile attachments. For a shape H, we say a system π uniquely assembles H if for all $A \in Prod_{\Gamma}$, there exists a $A \in Prod_{\Gamma}$ of shape H such that $A \to_{\pi} B$. An assembly sequence is a way to denote a particular possible sequence of growth for a given system and is formally defined to be any sequence of assemblies $(A_0, A_1, A_2, \ldots)$ such that $A_0 = S$ and $A_0 \to A_{(i+1)}$ for each i.

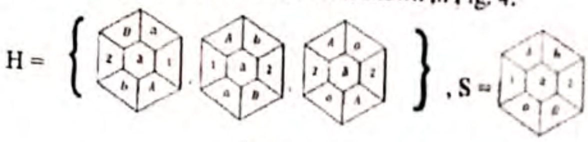
The terminal assemblies $Term_{\pi}$ is defined as $Term_{\pi} = \{A \in Prod_{\pi}, \nexists, B, s.t., A < B\}$. The produced assemblies include intermediate products of the self assembly process, whereas the terminal assemblies are just the end products and may be considered as the output.

If $A \in Prod_\pi \rightarrow \exists, B \in Term_\pi$ $s.t.A \rightarrow^*_\pi$, then π is said to be haltable. If π is haltable and $Term_\pi$ is finite, then π is said to be halting. A halting tile system uniquely produces C if $Term_\pi = C$. Note that if a tile system uniquely produces C then $Prod_\pi$ is a lattice. In general, if $Prod_\pi$ is a lattice, we say that π produces a unique pattern.

E. Example

Construct a $HTAS$ $\pi_1 = (\Sigma, \Gamma, H, S, I, F, \tau)$ where $\Sigma = \{a\}, \Gamma = \{a, b, A, B, 1, 2\}$, $I = (a, 1, a)$, $F = \{b, 2, B\}, \tau = 2$

Starting from the seed Assembly S in $HTAS$ a hexagonal tile from H is added if it makes any bond to a neighbour. Assembly proceeds by any sequence of single tile attachments from H as long as each tile attachment connects with strength $s(a,a) = s(b,b) = s(A,A) = s(B,B) = s(1,1) = s(2,2) = 2$ and the temperature τ=2. $HTAS$ recognizes the set of all parallelograms arrays of the form shown in Fig. 4.



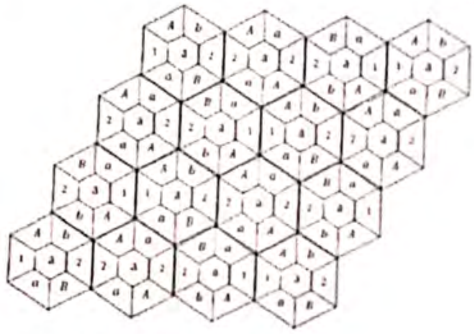


Fig. 4.: Parallelogram Arrays

F. *Theorem.*

$HTAS$ recognizes the language L_{spiral} of spiral picture in which the actions or transitions are labelled by six colors or denoted by the first letter of that color.

Proof:

Construct six Hexagonal Tile Assembly Systems $\Pi_P, \Pi_R, \Pi_Y, \Pi_G, \Pi_T$ and Π_B representing the automaton for the pink, red, yellow, green, turquoise and blue colored hexagonal patterns.

$\Pi_P = (\Sigma_P, \Gamma_P, H_P, S_P, I_P, F_P, \tau)$ where $\Sigma_P = \{Pink\}, \Gamma_P = \{a, A, 0, 1\}$, $I_P = F_P = \{0, a, A\}$, τ=1

$\Pi_R = (\Sigma_R, \Gamma_R, H_R, S_R, I_R, F_R, \tau)$ where $\Sigma_R = \{Red\}, \Gamma_R = \{a, B, 2, 3\}$, $I_R = F_R = \{2, a, B\}$, τ=1

$\Pi_Y = (\Sigma_Y, \Gamma_Y, H_Y, S_Y, I_Y, F_Y, \tau)$ where $\Sigma_Y = \{Yellow\}, \Gamma_Y = \{c, C, 2, 4\}$, $I_Y = F_Y = \{4, c, \}$, τ=1

$\Pi_G = (\Sigma_G, \Gamma_G, H_G, S_G, I_G, F_G, \tau)$ where $\Sigma_G = \{Green\}, \Gamma_G = \{b, C, 5, 6\}$, $I_G = F_G = \{5, b, C\}$, τ=1

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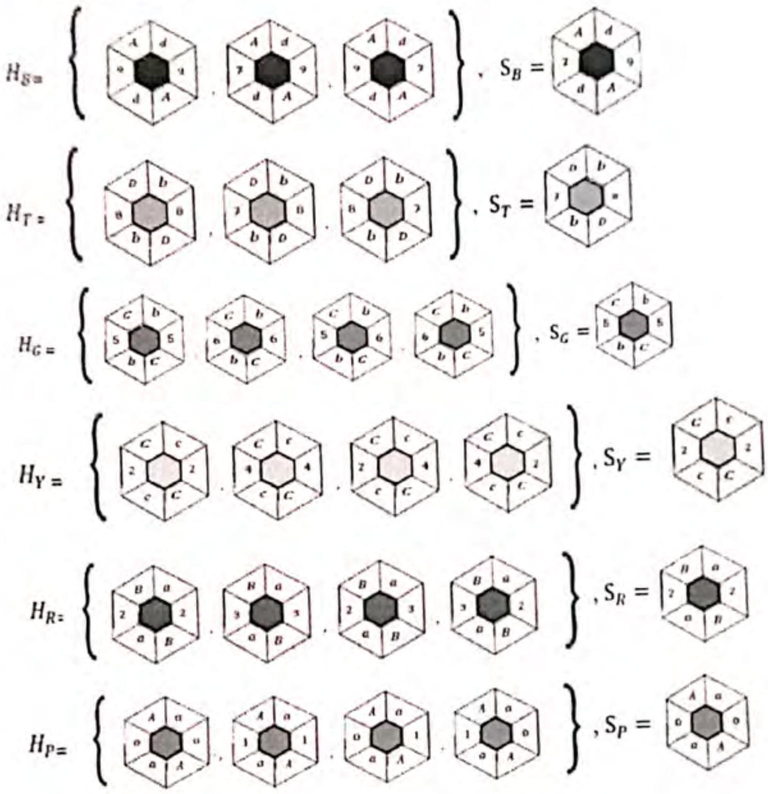
$\Pi_T = (\Sigma_T, \Gamma_T, H_T, S_T, I_T, F_T, \tau)$ *where* $\Sigma_T = \{Turquoise\}, \Gamma_T = \{b, D, 7, 8\},\ I_T = \ F_T = \{7, b, D\},\ \tau = 1$

$\Pi_B = (\Sigma_B, \Gamma_B, H_B, S_B, I_B, F_B, \tau)$ *where* $\Sigma_B = \{Blue\}, \Gamma_B = \{d, A, 7, 9\},\ I_B = \ F_B = \{7, d, A\},\ \tau = 1$

Now construct the Hexagonal tile assembly system Π_{spiral} as follows to generate the spiral image on six colors such that the six Assembling transitions can be continued from one transaction to another by their corresponding class or state transformation binding by their common borders. Terminal assemblies with suitable number of hexagonal patterns are taken from H_{spiral} and assembled in order of the colors and binding with equal strength with temperature equal to one.



$\Pi_{spiral} = (\Sigma_{spiral}, \Gamma_{spiral}, H_{spiral}, S_{spiral}, I_{spiral}, F_{spiral}, \tau)$ where

$\Sigma_{spiral} = \{\Sigma_P \cup \Sigma_R \cup \Sigma_Y \cup \Sigma_G \cup \Sigma_T \cup \Sigma_B\}$,

$\Gamma_{spiral} = \{\Gamma_P \cup \Gamma_R \cup \Gamma_Y \cup \Gamma_G \cup \Gamma_T \cup \Gamma_B\}$,

$H_{spiral} = \{Term_{\Pi_P} \cup Term_{\Pi_R} \cup Term_{\Pi_Y} \cup Term_{\Pi_G} \cup Term_{\Pi_T} \cup Term_{\Pi_B}\}$

$S_{spiral} = S_P, I_{spiral} = I_P, F_{spiral} = F_P$, and $\tau = 1$.

The generated Hexagonal tile assesmbly and the hexagonal spiral image after removing the side borders are shown in Fig. 5.

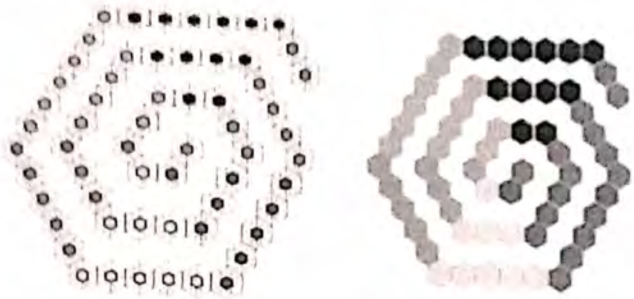


Fig. 5 : Hexagonal scenario and picture - spiral pattern

G. *Theorem.*

Sierpinski hexagon can be generated by $HTAS$.

Proof:

$\Sigma = \{black\}$, $\Gamma = \{a, b, A, B, 1, 2\}$, $I = F = \{a, 1, a\}$, $\tau = 1$

Starting from the seed Assembly S in $HTAS$, a hexagonal tile from H will be added if it makes any bond to a neighbour. Assembling proceeds by single tile attachments from H as long as each tile attachment connects with strength $s(a,a) = s(b,b) = s(A,A) = s(B,B) = s(1,1) = s(2,2) = 1$ and the temperature $\tau = 1$. The six tiles in H are attached in sequence as matching sides have positive strength and hence the Sierpinski hexagon (first iteration) is produced. The resulting pattern is denoted as $Term_{\pi_2}$.

Now the terminal assembly $Term_{\pi_2}$ is taken as the Produced assembly (seed assembly) to generate Sierpinski hexagon (second iteration). The terminal assembly is assembled with itself six times in sequence as matching sides have positive strength. Further, the resulting terminal assembly is added with itself to generate the next pattern. Continuing the above process repeatedly, $HTAS$ generates the Sierpinski hexagons. It recognizes the set of all Sierpinski hexagons as shown in Fig. 6.

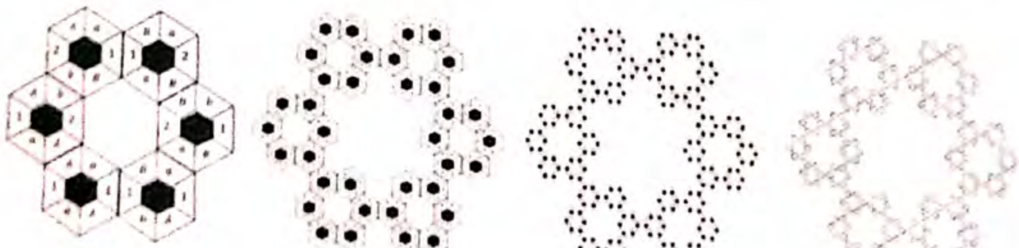


Fig. 6 : Sierpinski hexagons on four iterations