ON (1,2)γ-US SPACES

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Abstract

The aim of the paper is to introduce (1,2)γ-US space in a bitopological space and to derive a necessary and sufficient condition for a (1,2)γ-T₂ space.

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1. Introduction

In 1982, Mashhour et al.[8] defined pre-open sets in a topological space and Andrijevic [1] used the pre open sets to introduce γ-open sets in 1987 and showed that the set of all γ-open sets of a topological space is a topology in the same space. In 2004, Navalagi [9] investigated γ-separation axioms. In 2005, Ekici [3] introduced γ-*US* spaces. The concept of bitopological space was introduced by Kelly [4] in 1963. Soon after the introduction of bitopological spaces mathematicians concentrated on the extention of many topological concepts to bitopological spaces. Weak forms of open sets were introduced in bitopological spaces. A new class of sets called (1,2)α-open sets was defined by Lellis Thivagar et al. [6] and Lellis Thivagar and Athisaya Ponmani [7] introduced (1,2) pre-open set to define separation axioms and continuous functions in a bitopological space. Recently, the notion of (1,2)γ-open set was introduced by Karpagam and Athisaya Ponmani[5]

2. Preliminaries

Definition 2.1

A subset A of a space X is called pre-open [8], if $A \subset \text{int}\,cl(A)$.

The family of all pre-open sets of X is denoted by $PO(X)$. The complement of a pre-open set is defined to be pre-closed.

Definition 2.2

A subset A is called γ-open [1], if $A \cap S$ is pre-open, for all $S \in PO(X)$. The family of all γ-open sets in X is denoted by $\gamma O(X)$. The union of all γ-open sets of X contained in A is called γ-interior of A [1] and is denoted by $\gamma\text{-int}(A)$.

The complement of γ-open set is called γ-closed and is denoted by $\gamma C(A)$. The intersection of all γ-closed sets of X containing A is called γ-closure of A[1] and is denoted by $\gamma\text{-}cl(A)$.

Definition 2.3

A topological space X is called $\gamma - T_0$ [9] if for any distinct pair of points in X, there exists a γ-open set containing one of the points but not the other.

Definition 2.4

A space X is called $\gamma - T_1$ [9], if for each pair of distinct points x and y of X, there exists γ-open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

Definition 2.5

A space X is called $\gamma - T_2$ [9] if for each pair of distinct points x and y of X, there exists γ-open sets U and V such that $x \in U$, $y \in U$, $y \in V$ and $U \cap V = \phi$.

Definition 2.6

A space X is called $\gamma - R_1$ [3] if and only if for $x, y \in X$ with $\gamma - cl(\{x\}) \neq \gamma - cl(\{y\})$, there exist disjoint γ-open sets U and V such that $\gamma - cl(\{x\}) \subset U$ and $\gamma - cl(\{y\}) \subset V$.

Definition 2.7

Let X be a topological space and $S \subset X$. Then γ-kernel of S, denoted by γ-$ker(S)$ [2] and is defined to be the set γ-ker $(S) = \cap \left\{ U : U \in \gamma O(X),\ S \subset U \right\}$.

Definition 2.8

In a topological space X, a sequence (x_n) is called γ-converges to a point x of X, denoted by $(x_n) \xrightarrow{\gamma} x$ [3], if (x_n) is eventually in every γ-open set containing x.

Definition 2.9

A topological space x is called γ-US space [3] if every γ-convergent sequence (x_n) in X, γ-converges to a unique point.

Definition 2.10

Let X be a topological space. A subset F is called *sequentially γ-closed* [3] if every sequence in F γ-converging in X, γ-converges to a point in F.

Definition 2.11

Let X be a topological space X. A subset G of a space X is called sequentially γ-compact [2] if every sequence in G has a subsequence which γ-converges to a point in G.

Definition 2.12

If τ_1 and τ_2 are two topologies on a non-empty set X, then the triple (X, τ_1, τ_2) is called a bitopological space [4].

Definition 2.13

A subset A of X is called

i. $\tau_1\tau_2$-open [6] if $A \subset \tau_1 \cup \tau_2$

ii. $\tau_1\tau_2$-closed [6] if $A^C \subset \tau_1 \cup \tau_2$.

Definition 2.14

A subset A of the topological space X is called $(1,2)\gamma$-open [5] if $A \cap S$ is $(1,2)\gamma$-open, for all $S \in (1,2)PO(X)$.

The family of all $(1,2)\gamma$-open sets is denoted by $(1,2)\gamma O(X)$. The complement of $(1,2)\gamma$-open set is $(1,2)\gamma$-closed. The family of all $(1,2)\gamma$-closed sets is denoted by $(1,2)\gamma C(X)$.

Throughout this paper, X represents a bitopological spaces (X, τ_1, τ_2).

3. (1,2)γ-Separation Axioms

In this section, we introduce some separation axioms using $(1,2)\gamma$-open sets and carry out a comparative study of these axioms.

Definition 3.1

A bitpological space X is called $(1,2)\gamma$-T_0 if and only if for $x, y \in X$, $x \neq y$, there exist a $(1,2)\gamma$-open set containing only one of x and y but not the other.

Example 3.2

$$X = \{a,b,c\},\ \tau_1 = \{\phi,\{a\},X\},\ \tau_2 = \{\phi,\{b\},X\}$$

$(1,2)\gamma O(X) = \{\phi,\{a\},\{a,b\},X\}$. Then X is $(1,2)\gamma$-T_0.

Definition 3.3

A space X is called $(1,2)\gamma - T_1$ if and only if for $x, y \in X$, $x \neq y$, there exist $U, V \in (1,2)\gamma O(X)$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Example 3.4

$$X = \{a,b,c\},\ \tau_1 = \{\phi,\{a\},\{b\},\{a,b\},X\},\ \tau_2 = \{\phi,\{c\},\{a,c\},X\}$$

$(1,2)\gamma O(X) = \{\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\{b,c\},X\}$. Then X is $(1,2)\gamma - T_1$.

Remark 3.5

It is obvious that every $(1,2)\gamma - T_1$ space is $(1,2)\gamma - T_0$, but the converse is not true, in general.

Definition 3.6

A space X is called $(1,2)\gamma-T_2$, if for each pair of distinct points x and y of X, there exist $(1,2)\gamma$-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example 3.7

$X = \{a,b,c,d\}$,

$\tau_1 = \{\phi,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X\}$

$\tau_2 = \{\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},X\}$

$$(1,2)\gamma O(X) = \left\{\begin{array}{l}\phi,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\\ \{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X\end{array}\right\}.$$

Then X is $(1,2)\gamma-T_2$.

Remark 3.8

If a space X is $(1,2)\gamma-T_2$, then its $(1,2)\gamma-T_1$, but the converse is not true.

Theorem 3.9

A space X is $(1,2)\gamma-T_1$, if and only if the singletons are $(1,2)\gamma$-closed sets.

Proof

Assume X is $(1,2)\gamma-T_1$. Since the union of all the $(1,2)\gamma$-open sets not containing x which is $(1,2)\gamma$-open and its complement is $\{x\}$ in X. $\{x\}$ is $(1,2)\gamma$-closed. Hence each singleton set is $(1,2)\gamma$-closed.

Conversely, assume $\{x\}$ is $(1,2)\gamma$-closed in X. Let $x, y \in X$ with $x \neq y$. Then $X-\{x\}$ is a $(1,2)\gamma$-open in X which contains y and does not contain x and $X-\{y\}$ is a $(1,2)\gamma$-open in X which contain x and does not contain y. Therefore, X is a $(1,2)\gamma-T_1$ space.

Definition 3.10

Let S be a subset of X. The $(1,2)\gamma$-kernel of S, denoted by $(1,2)\gamma$-ker (S), is defined to be the set $(1,2)\gamma$-ker $(S) = \cap\left\{U \in \frac{(1,2)\gamma O(X)}{S} \subset U\right\}$.

Definition 3.11

A space X is called $(1,2)\gamma$-R_1, if for $x, y \in X$ with $x \notin (1,2)\gamma-cl(\{y\})$, there exist disjoint $(1,2)\gamma$-open sets U and V such that $x \in U$ and $y \in V$.

Theorem 3.12

A space X is $(1,2)\gamma$-R_1 if and only if $(1,2)\gamma$-ker $(\{x\}) \neq (1,2)\gamma-$ker $(\{y\})$ for $x, y \in X$, there exists disjoint $(1,2)\gamma$-open set U and V such that $(1,2)\gamma-cl(\{x\}) \subset U$ and $(1,2)\gamma-cl(\{y\}) \subset V$.

Proof

Let X be $(1,2)\gamma$ R_1. If $(1,2)\gamma$-ker $(\{x\}) \neq (1,2)\gamma-$ker $(\{y\})$, we have to prove that, there exist disjoint $(1,2)\gamma$-open sets U and V such that $(1,2)\gamma-cl(\{x\}) \subset U$ and $(1,2)\gamma-cl(\{y\}) \subset V$.

$(1,2)\gamma$-ker $(\{x\}) = \cap\left\{U \in \frac{(1,2)\gamma O(X)}{\{x\}} \subset U\right\}$. We have, $(1,2)\gamma$-ker $(\{x\}) \neq (1,2)\gamma$-ker $(\{y\})$. $(1,2)\gamma-cl(\{x\}) = \cap\left\{V \in \frac{(1,2)\gamma C(X)}{\{x\}} \subset V\right\}$. Since A is $(1,2)\gamma$-open if and only if A^C is $(1,2)\gamma$-closed. If $(1,2)\gamma$-ker $(\{x\}) \neq (1,2)\gamma$-ker $(\{y\})$ then $(1,2)\gamma-cl(\{x\}) \neq (1,2)\gamma-cl(\{y\})$. Since X is $(1,2)\gamma$-R_1, there exist disjoint open sets U and V such that $(1,2)\gamma-cl(\{x\}) \subset U$ and $(1,2)\gamma-cl(\{y\}) \subset V$.

Conversely, assume for $x, y \in X$ with $(1,2)\gamma$-ker$(\{x\}) \neq (1,2)\gamma$-ker $(\{y\})$, there exists disjoint $(1,2)\gamma$-open sets U and V such that $(1,2)\gamma-cl(\{x\}) \subset U$ and $(1,2)\gamma-cl(\{y\}) \subset V$.

$$(1,2)\gamma\text{-ker}(\{x\}) = \cap\left\{U \in \frac{(1,2)\gamma O(X)}{\{x\}} \subset U\right\}. \ (1,2)\gamma-cl(\{x\}) = \cap\left\{V \in \frac{(1,2)\gamma C(X)}{\{x\}} \subset V\right\}.$$

Since A is $(1,2)\gamma$-open if and only if A^C is $(1,2)\gamma$-closed. $(1,2)\gamma$-ker$(\{x\}) \neq (1,2)\gamma$-ker

$(\{y\}) \Rightarrow (1,2)\gamma - cl(\{x\}) \neq (1,2)\gamma - cl(\{y\})$. Therefore, we have for $x, y \in X$ with $(1,2)\gamma - cl(\{x\}) \neq (1,2)\gamma - cl(\{y\})$ then there exist disjoint $(1,2)\gamma$-open sets U and V such that $(1,2)\gamma - cl(\{x\}) \subset U$ and $(1,2)\gamma - cl(\{y\}) \subset V$. Hence X is $(1,2)\gamma - R_1$.

Theorem 3.13

A space X is $(1,2)\gamma - T_2$ if and only if is $(1,2)\gamma - R_1$ and $(1,2)\gamma - T_0$.

Proof

Let X be a $(1,2)\gamma - T_2$ space. Then X is $(1,2)\gamma - T_0$. Since X is $(1,2)\gamma - T_2$, by Theorem 3.9, $\{x\} = (1,2)\gamma - cl(\{x\}) \neq (1,2) - cl(\{y\}) = (\{y\})$, for $x, y \in X$, there exist disjoint $(1,2)\gamma$-open sets U and V such that $(1,2)\gamma - cl(\{x\}) \subset U$ and $(1,2)\gamma - cl(\{y\}) \subset V$. Thus X is a $(1,2)\gamma - R_1$ space. Hence X is $(1,2)\gamma - R_1$ and $(1,2)\gamma - T_0$ space.

Conversely, let X be $(1,2)\gamma - R_1$ and $(1,2)\gamma - T_0$. Let x, y be any two distinct points of X. Since X is $(1,2)\gamma - T_0$, then there exist a $(1,2)\gamma$-open set U such that $x \in U$ and $y \notin U$ or there exist a $(1,2)\gamma$-open set V such that $y \in V$ and $x \notin V$. Let $x \in U$ and $y \notin U$. Then $y \notin (1,2)\gamma\text{-ker}(\{x\})$ and then $(1,2)\gamma\text{-ker}(\{x\}) \neq (1,2)\gamma\text{-ker}(\{y\})$. Since X is $(1,2)\gamma - R_1$, there exist disjoint $(1,2)\gamma$-open sets U and V such that $x \in (1,2)\gamma - cl(\{x\}) \subset U$ and $y \in (1,2)\gamma - cl(\{y\}) \subset V$.

Therefore, X is $(1,2)\gamma - T_2$.

4. $(1,2)\gamma - US$ Space

Definition 4.1

A sequence (x_n) is called $(1,2)\gamma$-converges to a point x of X, denoted by $(x_n) \xrightarrow{(1,2)\gamma} x$, if (x_n) is eventually in every $(1,2)\gamma$-open set containing x.

Definition 4.2

A space X is called $(1,2)\gamma-US$ space if every $(1,2)\gamma$-convergent sequence (x_n) in X $(1,2)\gamma$-converges to a unique point in X.

Definition 4.3

A set F is called sequentially $(1,2)\gamma$-closed if every sequence in F $(1,2)\gamma$-converging in X $(1,2)\gamma$-converges to a point in F.

Definition 4.4

A subset G of a space X is called sequentially $(1,2)\gamma$-compact if every sequence in G has a subsequence which $(1,2)\gamma$-converges to a point in G.

Theorem 4.5

Every $(1,2)\gamma-T_2$ space is $(1,2)\gamma-US$.

Proof

Let X be $(1,2)\gamma-T_2$ space and (x_n) be a sequence in X. Suppose that (x_n) $(1,2)\gamma$-converges to two distinct points x and y. That is (x_n) is eventually in every $(1,2)\gamma$-open set containing x and also in every $(1,2)\gamma$-open set containing y, a contradiction. Hence the space X is $(1,2)\gamma-US$.

Theorem 4.6

Every $(1,2)\gamma-US$ space is $(1,2)\gamma-T_1$.

Proof

Let X be $(1,2)\gamma-US$ space. Let x, y be two distinct points of X. Consider the sequence (x_n), where $x_n = x$, for all n. Clearly, $(x_n)\gamma$-coverges to x. Also since $x \neq y$ and X is $(1,2)\gamma-US$, (x_n) cannot $(1,2)\gamma$-converge to y. That is, there exists a $(1,2)\gamma$-open set V containing y, but not x. Similarly, if we consider the sequence (y_n), where $y_n = y$ for all n. Proceeding as above, we get a $(1,2)\gamma$--open set U containing x, but not y. Thus the space X is $(1,2)\gamma-T_1$.

Theorem 4.7

In a (1,2)γ-*US* space, every sequentially (1,2)γ compact set is sequentially (1,2)γ-closed.

Proof

Let *X* be (1,2)γ-*US* space. Let *Y* be a sequentially (1,2)γ compact subset of *X*. Let (x_n) be a sequence in *Y* which is (1,2)γ-convergent to a point in *Y*. Let (x_{n_k}) be a subsequence of (x_n) which is (1,2)γ-converges to a point $y \in Y$. Also a subsequence (x_{n_k}) of (x_n), (1,2)γ-converges to $x \in X$. Since (x_{n_k}) is a sequence in the (1,2)γ-*US* space *X*, $x = y$. Thus *Y* is sequentially (1,2)γ-closed set.

Theorem 4.8

Every (1,2)γ-open set of a (1,2)γ-*US* space is (1,2)γ-*US* space.

Proof

Let *X* be a (1,2)γ-*US* space and $Y \subset X$ be a (1,2)γ-open set. Let (x_n) be a sequence in *Y*. Suppose that (x_n) (1,2)γ-converges to *x* and *y* in *Y*. We shall prove (x_n) (1,2)γ-converges to *x* and *y* in *X*. Let *U* be any (1,2)γ-open subset of *X* containing *x* and *V* be any (1,2)γ open set of *X* containing *y*. Then $U \cap Y$ and $V \cap Y$ are (1,2)γ-open in *Y*. Therefore, (x_n) is eventually in $U \cap Y$ and $V \cap Y$ and so in *U* and *V*. Since *X* is (1,2)γ-*US*, this implies that $x = y$. Hence the subspace *Y* is (1,2)γ-*US*.

5. References

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