ACYCLIC EDGE COLOURINGS OF PLANAR GRAPHS

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Abstract

A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called edge-colouring. The acyclic chromatic index of G, denoted χ'a(G), is the minimum number k such that G has a proper edge k-colouring. Acyclic Edge colouring conjecture (the AECC for short), which states that χ'a(G) ≤ Δ(G)+2 for all graphs G.I. Fiamcik proved in [8] that Δ(G). (Δ(G) - 1) + 1 is an upper bound for the acyclic chromatic index of a graph G and conjectured that χ'a(G) < Δ(G)+2. In [1] Alon et al. presented a linear upper bound on χ'a(G). They proved that χ'a(G) ≤ 64Δ(G), which was later improved to 16Δ(G) by Molloy and Reed [10]. In this paper we provide new upper bounds for the acyclic chromatic index for the classes of planar graphs, 3-fold graphs, triangle-free planar graphs and 2-fold graphs.

Keywords: *edge – colouring, planar graphs, chromatic index*

1. INTRODUCTION

All graphs which we consider are finite and simple. For any graph G, we denote its vertex set, edge set, maximum degree and minimum degree by V(G), E(G), Δ(G) and δ(G), respectively. For a vertex v, its degree is denoted by dG(v) or simply d(v) when no confusion can arise. For undefined concepts we refer the reader to [7].As usual [k] stands for the set {1,..., k}. A mapping C: E(G)→[k] is called a proper edge k-colouring of a graph G provided any two adjacent edges receive different colours. A proper edge k-colouring C of G is called an acyclic edge k colouring of G if there are no bichromatic cycles in G under the colouring C. in other words, for every pair of distinct colours i and j, the subgraph induced in G by all the edges which have either colour i or j, is acyclic. The smallest number k of colours such that G has an acyclic edge k-colouring is called the acyclic chromatic index of G and is denoted by $\chi'a(G)$.

$\chi'(G)$ stands here for the chromatic index of G, defined as the minimum number k such that G has a proper edge k-colouring. The notion of acyclic colouring of a graph was introduced by Grunbaum [9] for vertex colouring and it was later extended to the edge version as well. Fiamcik proved in [8] that $\Delta(G).(\Delta(G)-1)+1$ is an upper bound for the acyclic chromatic index of a graph G and conjectured that $\chi'a(G) \leq \Delta(G)+2$. In [1] Alon et al. presented a linear upper bound on $\chi'a(G)$. They proved that $\chi'a(G) \leq 64\Delta(G)$, which was later improved to $16\Delta(G)$ by Molloy and Reed [10].

In [2] Alon et al. independently made the Acyclic Edge colouring conjecture (the AECC for short), which states that $\chi'a(G) \leq \Delta(G)+2$ for all graphs G. In [2] this conjecture was proved to be true for random d-regular graphs (asymptotically) and for graphs having large girth.

The AECC was also verified for some special classes of graphs including subcubic graphs [8,13], outerplanar graphs [11] and grid-like graphs [12]. For the last two classes of graphs the better bound $\Delta(G)+1$ was obtained. In [13] Skulrattanakulchai presented a linear algorithm to acyclically colour, with 5 colours, the edges of any cubic graph. Very recently Basavaraju and Chandran proved in [4], that $\chi'a(G) \leq 4$ for any non-regular connected sub-cubic graph G. They also showed in [5] that $\chi'a(G) \leq 7$ for any graph G of maximum degree 4 and in [6] that $\chi'a(G) \leq \Delta(G)+1$ for any 2-degenerated graph G. The problem of determining the acyclic chromatic index of a graph is NP-complete, what was shown by Alon and Zaks in [3]. The authors also presented a polynomial algorithm to acyclically colour the edges of a graph G with $\Delta(G)+2$ colours, provided the length of the shortest cycle in G is greater that $c(\Delta(G))3$, for some constant c.

A graph G is called k-fold if the edges of G can be partitioned into k forests. And start by introducing some necessary notations.

Let G be a graph and $v \in V(G)$. We call v a k-vertex if $dG(v) = k$. Similarly, v is called a k^- - or a k^+ - vertex if its degree is at most k or at least k, respectively.

We denote by lk(v) (respectively, lk⁻(v), lk⁺(v)) the cardinality of the set of those neighbours of v, which are k-vertices (respectively, k⁻ -vertices, k⁺ - vertices).

2. PLANAR GRAPHS WITH GIRTH 7

Claim: 2.1

There is no 2-vertex in v in G incident with vertices v1 and v2 such that d(v1) + d(v2) ≤ Δ+1.

Claim: 2.2

Let u and v be a pair of sub adjacent vertices. If d(v) < Δ, then the number of 2 vertices adjacent to v is at most d(u)+d(v) - Δ - 1.

Claim: 2.3

Let v be a Δ-vertex subadjacent to a vertex u in G. Then the number of 2-neighbors of v is at most $d(u)$.

Claim: 2.4

There is no vertex v in G with d(v) < Δ adjacent only to 2-vertices.

Lemma: 2.1

Let Δ ≥ 6. Every planar graph with girth at least 6 and maximum degree at most admits an acyclic edge colouring with colours.

Lemma: 2.2

Let Δ ≥ 5. Every planar graph with girth at least 7 and maximum degree at most admits an acyclic edge colouring with colours.

Suppose G is a minimal counte rexample to Lemma 2.2. If Δ ≥ 6, then Lemma 2.2 follows from Lemma 2.1. Therefore, we may assume that Δ = 5 and Δ(G) ≤ 5.

2.1 Discharging rules

Let the initial charge be set as follows:

- $w(v) = 5d(v) - 14$ for each vertex v of G;
- $w(f) = 2d(f) - 14$ for each face f of G.

By Euler's formula we have that the sum of charges of vertices and faces are −28. It is clear that since $g \geq 7$ all the faces have nonnegative charge. Vertices of degree 5 have charge 11, vertices of degree 4 have charge 6, vertices of degree 3 have charge 1, and vertices of degree 2 have charge −4.

We redistribute the charge among vertices by the following rules:

(R1) Let v be a 2-vertex with neighbours $v1$ and $v2$ such that $d(v1) \leq d(v2)$.

(R1a) If $d(v1) = 2$, then v sends 0 of charge to $v1$ and −4 of charge to $v2$.

(R1b) If $d(v1) = 3$, then v sends -1/3 of charge to $v1$ and it receives -11/3 of charge to $v2$.

(R1c) If $d(v1) \geq 4$, then v sends −2 of charge to both, $v1$ and $v2$.

Since $\Delta = 5$, by Claim 1 for each 2-vertex with neighbours with degrees $d1$ and $d2$ we have $d1 + d2 \geq \Delta + 2 = 7$. It is easy to see that 2-vertices send all their negative charge to their neighbours of degree at least 3.

2.1.1 3-vertices

Let v be a 3-vertex in G. Its initial charge is 1. By *(R1b)* it receives -1/3from each its 2-neighbour, hence its charge is at least 1-3*(1/3) =0.

2.1.2 4-vertices

Let v be a 4-vertex in G. Its initial charge is 6. If it has no 2-neighbours, its charge does not change. By Claim 2.1 it cannot be sub adjacent to a 2-vertex. If it is sub adjacent to a 3-vertex, by Claim 2. 2 the number of 2-neighbors of v is at most $3 + 4 - \Delta - 1 = 1$, hence, it has only one 2-neighbor from which it receives −11/3 of charge by *(R1b)*. Its charge is clearly nonnegative. If v is not sub adjacent to any $\leq$ 3-vertex, then by Claim 2.4 it can have at most three 2 neighbours, from which it receives −2 of charge by *(R1c)*. Its charge is (at least) $6 - 3 * 2 = 0$.

2.1.3 5-vertices

Let v be a 5-vertex in G. Its initial charge is 11. If it has no 2-neighbors, its charge does not change.

If v is sub adjacent to a 2-vertex, then by Claim 3 it has at most two 2-neighbors, which send at most −4 of charge each. The charge of v is at least

$11 - 2 * 4 = 3 \geq 0$. If v is not sub adjacent to any 2-vertex and v is sub adjacent to a 3-vertex, it has at most three 2-neighbours, which send at most $-11/3$ of charge each. The charge of v is at least $11 - 3 * (11/3) = 0$.

If v is not sub adjacent to any ≤ 3-vertex, then all its 2-neighbors send -2 of charge by (R1c), the charge of v is at least $11 - 5 * 2 = 1 \geq 0$.

All the vertices of G have nonnegative charge, a contradiction which establishes the lemma.

LEMMA 2.3

Let G be a graph such that $|E(G)| \leq 2|V(G)| - 1$ and $\delta(G) \geq 2$. Then G contains at least one of the following configurations:

(A1) a 2-vertex adjacent to a 5^--vertex,

(A2) a 3-vertex adjacent to at least two 5^--vertices,

(A3) a 6-vertex adjacent to at least five 3^--vertices,

(A4) a 7-vertex adjacent to seven 3^--vertices or

(A5) a vertex x such that at least $d(x) - 3$ of its neighbours are 3^--vertices, and moreover one of them is of degree 2.

Proof

We use the discharging method to prove the lemma. Let $G = (V, E)$, $\delta(G) \geq 2$ and $|V| = n$, $|E| = m$. We have $m \leq 2n - 1$. Initially, we will define a mapping f on the set of vertices of G as follows: for each $x \in V$ let $f(x) = d(x) - 4$. It is easy to observe that $\sum_{x \in V} f(x) \leq -2$ which follows from the inequality $m \leq 2n - 1$. In the discharging step, we redistribute the values of f between adjacent vertices according to the two rules described below to obtain the function f'.

- If x is a 5^--vertex, then x does not give anything to its neighbours.
- If x is a 6^+-vertex, then x gives 1 to each 2-vertex in its neighbourhood and ½ to each 3-vertex in its neighbourhood.

After this procedure, each vertex x has a new value $f'(x)$. But since the charges were only re-distributed, the sums of values of the functions f' and f, counting over all the vertices remain the same. We now show that if G does not contain any of they configurations (A1)-(A5), then for each vertex x, the value $f'(x)$ will be nonnegative. This leads us to a contradiction with the fact that $\sum_{x \in V} f'(x) \sum_{x \in V} f(x) \leq -2$.

To calculate the values $f'(x)$, we consider a number of cases, depending on d(x).

- If d(x) = 2, then $f'(x)$ = -2+1.16 +(x) = -2+2=0, because (A1) does not hold.
- If d(x) = 3, then $f'(x)$ = -1+(1/2).16 +(x) ≥ -1+(1/2).2 = 0, because (A2) does not hold.
- If d(x) = 4, then $f'(x) = f(x) = 0$.
- If d(x) = 5, then $f'(x) = f(x) = 1$.
- If d(x) = 6, then $f'(x)$ = 2-1. 12(x) = -1/2. 13(x). Let us initially assume that x has at least one neighbour of degree 2. Now, since (A5) does not occur, x can have at most d(x) – 4 = 6 – 4 = 2 neighbours of degree at most 3, and hence $f'(x) \geq 2 - 2 = 0$. For the case when x does not have any neighbour of degree 2, one can observe that $f'(x)$ = 2-(1/2). 13(x) ≥ 2 – (1/2).4 = 0, because (A3) does not hold.
- If d(x) = 7, then $f'(x)$ = 3 - 1. 12(x) – (1/2). 13(x). First we will assume that x has at least one neighbour of degree 2. Then because (A5) does not occur, we notice that x can have at most 3 and therefore $f'(x) \geq 3 - 3 = 0$. In the other case where x does not have a degree 2 neighbour, we have $f'(x)$ = 3-(1/2) . 13(v) ≥ 3 - (1/2).6 = 0, since (A4) does not hold.
- If d(x) ≥ 8, then if x has at least one neighbour of degree 2 and (A5) does not hold, x can have at most d(x)-4 neighbours of degree at most 3 and therefore $f'(x) \geq 0$. For the case when x does not have a neighbour of degree 2, we have $f'(x)$=d(x)-4-(1/2).13(x)≥d(x)-4-.d(x)=.d(x)-4≥0.

Since for each vertex x of G the value $f'(x)$ is nonnegative, we obtain a contradiction with the inequality (1).

3. Acyclic k –Colouring Graphs

In the sequel we frequently use the following notations.

Let C be an acyclic k-colouring of a graph G. For any vertex v, denote by C(v) the set of colours assigned by C to the edges incident to v. for $W \subseteq V(G)$ we define C(W) = Uw C(w). For an edge uv, C(uv) is the colour of uv in C.

If v and u are two distinct vertices of G then let WG (v,u) stand for the set of neighbours w of the vertex v in G for which C(vw) ∈ C(u). Notice that the order of v and u is important here and that the set WG (v,u) could be empty.

LEMMA 3.1

Let G be a graph, uv ∈ E(G) and let C be an acyclic k-colouring of G - uv. If |C(v) UC(u) UC(WG-vu(V,U))| < k, then the colouring C can be extended to an acyclic k-colouring of G.

Proof

It is enough to colour the edge uv with any colour α from the set [k] - (C(v) UC(u) UC(WG-vu(V,U))), to obtain an acyclic k colouring of G.

LEMMA 3.2

If G is a graph such that $|E(G')| \leq 2|V(G')|-1$ for each $G' \subseteq G$, then $\chi'a(G) \leq \Delta(G) + 6$.

We may assume without loss of generality that H is 2-connected. Otherwise we can obtain an acyclic k-colouring of each its 2-connected component and combine them (by renaming some colours if needed) to get an acyclic k-colouring of the entire graph.

Hence we have $\delta(H) \geq 2$ and, by Lemma 2.3, the graph H contains at least one of the configurations (A1)-(A5).

According to Lemma 3.1, it is sufficient to show that there exists an edge vu and an acyclic k-colouring C of H-vu such that |C(v) C(u) C(WH-vu(v,u))|<k. we will consider a number of cases, depending on which of the configurations (A1)-(A5) occurs in H. In each case will point out such an edge which we can use with Lemma 3.1 to obtain a contradiction.

CONFIGURATION (A1)

If H contains a 2-vertex x adjacent to a 5-vertex y, then let z be the remaining neighbour of x. Moreover, let H' = H-xz. Clearly, $\chi'a(H') \leq K$. let C be any acyclic k-colouring of H' and assume that C(x)=a. We consider two cases.

Case 1. If |C(z)∩{a}|=0 then |C(z)∩{a}|≤Δ(H) and WH'(x,z)=Ø. Therefore, from Lemma 3.1, H has an acyclic k-colouring, a contradiction.

Case 2. If |C(z)∩{a}|=1 then WH'(x,z)={y}. from the fact dH(y) ≤ 5 we have |C(y)| ≤ 5, therefore |C(x) C(z) C(y)| ≤ Δ(H)+3. By Lemma 3.1, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

CONFIGURATION (A2)

If there is a a 3-vertex x in H adjacent to two 5 - vertex z and z_1, then let y be the third neighbour of x. Moreover let H' = H-xz. Since H' has less edges than H, χ'a(H') ≤ K. Let C be an acyclic k-colouring of H'. We consider the following cases.

Case 1. |C(z)∩C(x)|≤1. We have |C(x) C(z) C(WH'(x,z)|≤Δ(H)+4 and by Lemma 3.1, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

Case 2. |C(z)∩C(x)|=2.

Subcase 2.1. If C(x,y) > C(z1), then |C(z) C(z1) C(x)| ≤ Δ(H)+3. Therefore we can recolour (in H') the edge xz1 with a colour ∝ > C(z) C(z1) C(x), obtaining an acyclic k-colouring C' of H' reducing it to the previous case.

Subcase 2.2 C(xy) ∈ C(z1).

Case 2.2(a). If C(xz1) > C(y), then |C(x) C(y) C(z)|≤Δ(H)+3. Thus we can recolour, in H', the edge xy with a colour ∝ > C(x)C(y) C(z) to obtain an acyclic k- colouring C' of H' and we are in the first case.

Case 2.2(b). If C(xz1) ⊂ C(y), then |C(x) C(z) C(z1) C(y)|≤Δ(H)+5 and, by lemma 2.3, we can extend the colouring C to an acyclic k-colouring of H, a contradiction.

CONFIGURATION (A3)

If in H, there is a 6-vertex x adjacent to five 3 - vertices say z, z_1, z_2, z_3 and z_4, then let y be the remaining neighbour of x. Moreover let H' = H-xz. Since H' has less edges than H, χ'a(H') ≤k. Let C be an acyclic k-colouring of H'.

Case 1. |C(x)∩C(z)| ≤ 1. It follows |C(x)C(z) C(WH'(x,z))| ≤ Δ(H) + 5 and hence, by Lemma 3.1, we extend the colouring C to an acyclic k-colouring of H, a contradiction.

Case 2. |C(x)∩C(z)| = 2 and C(xy) > C(z). Clearly, |C(x)C(z) C(WH'(x,z))| ≤ 9. Therefore, according to Lemma 3.1 and since Δ(H) ≥ 6, H has an acyclic k-colouring, a contradiction.

Case 3. |C(x)∩C(z)| = 2 and C(xy) ϵ C(z). Assume without loss of generality that C(xz1) ⊂ C(z).

Subcase 3.1 If |C(x)∩C(z)| = 1, then we recolour the edge xy (in H') with a colour ∝ > C(x) C(y) to obtain an acyclic k-colouring C' and we are in the previous case.

Subcase 3.2 If |C(x)∩C(z)| ≥ 2, then |C(x) C(z) C(WH'(x,z))| ≤ Δ(H) + 5 and, by Lemma 3.1, we can extend the colouring C to an acyclic k-colouring of H again a contradiction.

CONFIGURATION (A4)

If H has a 7-vertex x adjacent to seven 3 -vertices, then let z be one of its neighbours and let H' = H-xz. Since H' has less edges than H, χ'a(H') ≤ k. Let C be an acyclic k-colouring of H'. We can observe that |C(x) C(z) C(WH'(x,z))| ≤ 10. Therefore, according to Lemma 3.1 and since Δ(H) ≥ H has an acyclic k-colouring, a contradiction.

CONFIGURATION (A5)

If none of the cases (A1)-(A4) occurs, then there must be a vertex x in H such that at least dH(x)-3 of its neighbours are 3-vertices and one of them, say z, is of degree 2. Let us consider the graph H' = H – xz. Since H' has less edges than H, χ'a(H') ≤ k. Let C be an acyclic k-colouring of H'. Let C(z) = {C(xy): y ∈ NH(x)-{z} and dH(y) ≤ 3}.

Case 1. If a > C1, then |C(x)C(z) C(WH'(x,z))|≤Δ(H)+1. Therefore, by Lemma 3.1, H has an acyclic k-colouring, a contradiction.

Case 2. On the other hand, if a ϵ C1, then let y be the neighbour of z in H'. there is a colour ∝ > C1 C(y) such that we can recolour (in the graph H') the edge zy with ∝, obtaining in this way an acyclic k-colouring C' of H' and we will fail in the previous case.

We can now formulate two theorems, which provide upper bound on the acyclic chromatic index for 2-fold graphs and the class of planar graphs without cycles of length three.

THEOREM 3.1

Let G be any 2-fold graph. Then, $\chi'a(G) \leq \Delta(G) + 6$.

Proof

It follows from Lemma 3.2 and the two known facts:

(i) Any subgroup of a 2-fold graph is also 2-fold;

(ii) In any 2-fold graph G, of order at least 2, $|E(G)| \leq 2|V(G)| - 2$.

THEOREM 3.2

Let G be any planar graph without cycles of length three. Then $\chi'a(G) \leq \Delta(G) + 6$.

Proof

It is known that, $|E(G)| < 2|V(G)| - 4$ for any planar graph G without cycles of length three having order at least 3. This can be proved using Euler's formula. Clearly, this property holds for any subgraph also. Therefore, from Lemma, $\chi'a(G) \leq \Delta(G) + 6$.

LEMMA .3.3

Let G be any graph such that $|E(G)| < 3|V(G)| - 1$ and $\delta(G) \geq 3$. Then at least one of the following configurations occurs in G:

B1) a 3-vertex adjacent to an 11^--vertex,

B2) a 4-vertex adjacent to at least two 11^--vertices,

B3) a 5-vertex adjacent to at least three 11^--vertices,

B4) a vertex x such that $12 \leq d(x) \leq 14$ and at least $d(x) - 2$ of its neighbours are 5^--vertices,

B5) a 15-vertex adjacent to at least fourteen 5^--vertices,

B6) a vertex x such that $16 \leq d(x) < 17$ and all its neighbours are 5^--vertices, (B7) a vertex x such that at least d(x)-5 of its neighbours are 5^--vertices and at least one of them is of degree 3.

Proof

Let $G = (V,E)$ be such that $\delta(G) \geq 3$ and $m \leq 3n - 1$, where $|V| = n$, $|E| = m$. First, we define a function f on V as follows: for each $x \in V$ let $f(x) = d(x)$-6. Clearly, $\sum_{x \in V} f(x) \leq -2$, which follows from the inequality $m \leq 3n - 1$. In the next step we will distribute the values of f between adjacent vertices, according to the two rules described below to obtain the function f'.

- If x is a 11^--vertex, x does not give anything to its neighbours.
- If x is a 12^+-vertex, then x gives 1 to each 3-vertex in its neighbourhood, 2/3 to each 4-vertex in its neighbourhood and 1/3 to each 5-vertex in its neighbourhood. Now each vertex x has a new value f'(x), but the sums of values of the functions f' and f, counting over all the vertices, remain the same.

In the following we show that, if G does not contain any of the configurations (B1)-(B7), then f'(x) will be nonnegative for each x, which will lead us to a contradiction with the fact that $\sum_{x \in V} f'(x) \sum_{x \in V} f(x) \leq -2$.

We consider a number of cases depending on the degree of x.

- If d(x) = 3, then f'(x) = -3+1. $L12_+(x)$ = -3 + 3 = 0, because (B1) does not hold.
- If d(x) = 4, then f'(x) = -2+(2/3). $L12_+(x) \geq -2 + (2/3).3 = 0$, because (B2) does not hold.
- If d(x) = 5, then f'(x) = -1 + (1/3).$l12_+(x) \geq -1 + (1/3).3 = 0$, because (B3) does not hold.
- If $6 \leq d(x) \leq 11$, then $f'(x) = f(x) \geq 0$.
- If $12 \leq d(x) \leq 14$, then f'(x) = d(x) – 6 – 1.l3(x) – (2/3).l4(x)-(1/3).l5(x). If we assume that x has at least one neighbour of degree 3, then since (B7) does not occur, x can have at most d(x)-6 neighbours which are 5^--vertices, and hence $f'(x) \geq 0$. If the case is that none of the neighbours of x is a 3-vertex,

we observe that $f(x) \geq d(x)$-6-(2/3).l_5-(x) $\geq d(x)$-6-(2/3).(d(x)-9) > 0, since (B4) does not hold.

- When $d(x) = 15$, $f(x) = 9$-l_3(x)-(2/3).l_4(x)-(2/3).(d(x)-3) ≥ 0, since (B4) does not hold
- When $d(x) = 15$, $f(x) = 9$-l_3(x)-(2/3).l_4(x)-(1/3).l_5(x). If we assume that x is adjacent to a 3-vertex, then from the fact that (B7) is absent, we can see that x can be adjacent to at most d(x)-6 vertices of degree at most 5 implying $f(x) \geq 0$. If x is not adjacent to a 3-vertex, then $f(x) \geq 9$-(2/3).l_5-(x) ≥ 9-(2/3).13 ≥ 0, because (B5) does not hold.
- If $16 \leq d(x) \leq 17$, then $f(x) = d(x)$-6-l_3(x)-(2/3).l_4(x)-(1/3).l_5(x). If we assume that x is adjacent to a 3-vertex, then since (B7) does not occur we know that x can be adjacent to at most d(x)-6 vertices of degree at most 5 and $f(x) \geq 0$. On the other hand, if x is not adjacent to a 3-vertex then $f(x) \geq$ d(x)-6-(2/3).l_5-(x) $\geq$ d(x)-6-(2/3).(d(x)-1) ≥ 0, since (B6) does not occur
- When $d(x) \geq 18$, $f(x) = d(x)$-6-l_3(x)-(2/3).l_4(x)-(1/3).l_5(x). Supposing that x is adjacent to a 3-vertex and since (B7) does not occur, x can be adjacent to at most d(x)-6 vertices of degree at most 5 and $f(x) \geq 0$. Again, if x is not adjacent to a 3-vertex then $f(x) \geq d(x)$-6-(2/3).l_5(x) $\geq$ d(x)-6-d(x).(2/3) ≥ 0.

Since for every vertex of G the value f(x) is nonnegative, we obtain a contradiction with the inequality (2).

LEMMA 3.4

If G is a graph such that $|E(G')| \leq 3|V(G')| - 1$ for each $G' \subseteq G$, then $\chi'a(G) \leq 2\Delta(G) + 29$.

Proof

As in proof of Lemma 3.2 above, assume that H is a minimal counter example to Lemma 3.4 (with the number of edges as least as possible).

Let k stands for $2\Delta(H) + 29$. Once again we can assume without loss of generality that H is 2-connected and hence $\delta(H) \geq 2$. One can observe that if $\delta(H) = 2$ then $\chi'a(H) \leq k$. Clearly, if e is an edge incident to a 2-vertex, then there is an acyclic k-colouring C of H – 3. We are using more than 2Δ(H) colours, therefore this colouring can be easily extend to an acyclic k-colouring of H.

Hence δ(H) ≥ 3. Then, by Lemma 3.3, H contains at least one of the configurations (B1)-(B7).

According to Lemma 3.1, it is enough to show that |C(v) C(u) C(WH-vu(v,u))|< k for some edge vu and an acyclic k-colouring C of H-vu. We will consider different case depending on δ(H) and which of the configurations (B1)-(B7) occurs in H. As in proof of Lemma 3.2 we will try to find a suitable edge in order to use Lemma 3.1 and make a contradiction with the fact that H is a minimal counterexample.

CONFIGURATION (B1)

If there is a 3-vertex x, adjacent to a 11- -vertex y in H, then assume that z is another neighbour of x (in H) and let H' = H-xz. From the fact that H is a minimal counterexample we see that H' has an acyclic k-colouring, say C, moreover, since dH(y) ≤ 11, we have |C(z) C(x) C(WH(x,z))| ≤ 2Δ(H)+8. According to Lemma 3.1, it follows that H has an acyclic k-colouring. Moreover, because dH(y1), dH(y2) ≤ 11, we have |C(z) C(x) C(WH'(x,z))| ≤ 2Δ(H)+18. From Lemma 3.1 it follows that H has an acyclic k-colouring, a contradiction.

CONFIGURATION (B2)

If H contains a 4-vertex x adjacent to at least two 11^- -vertices say y_1, y_2, let z be any other neighbour of x (in H) and let H' = H-xz. Since H is a minimal counterexample, H' has an acyclic k-colouring. Moreover, because dH(y_1), dH(y_2) ≤ 11. We have |C(z) C(x) C(WH'(x,z))| ≤ 2Δ(H)+18. From Lemma 3.1 it follows that H has an acyclic k-colouring, a contradiction.

CONFIGURATION (B3)

If in H there is a 5- vertex x adjacent to at least three 11^- - vertices y_1, y_2, y_3, then let z be any other neighbour of x. Let H' = H – xz. As before, H' has an acyclic k-colouring C. We also have dH(y1), dH(y2), dH(y3) ≤ 11 implying |C(z)

C(x) C(WH'(x,z))| ≤ max {Δ(H) + 30, 2Δ(H) + 28} ≤ 2Δ(H)+28, because Δ(H) ≥ 5. As in the previous case, it follows that H has an acyclic k-colouring, a contradiction.

CONFIGURATION (B4)

If H has a vertex x of degree 12 or 13 or 14, adjacent to at least dH(x)-2 vertices of degree at most 5 then let z be any of them and let H' = H-xz. From the fact that H is a minimal counterexample we have that H' has an acyclic k-colouring C. From the fact that all, except at most two, neighbours of x in H are of degree at most 5 we have |C(z) C(x) C(WH'(x,z))| ≤ 2Δ(H)+19. According to Lemma 3.1, it follows that H has an acyclic k-colouring, a contradiction.

CONFIGURATION (B5)

If there is a 15-vertex x adjacent to atleast fourteen 5 - vertices, then let z be any of them and let H' = H-xz. As before H' has an acyclic k-colouring C. we notice that |C(z) C(x) C(WH'(x,z))| ≤ Δ(H) + 25 ≤ 2Δ(H) + 10, because Δ(H) ≥ 15. It follows from Lemma 3.1 that H has an acyclic k-colouring, a contradiction.

CONFIGURATION (B6)

If H contains a vertex x of degree either 16 or 17, such that all its neighbours are 5 -vertices, then let z be any of this neighbours and let H' = H-xz. From the fact that H is a minimal counterexample we have that H' has an acyclic k-colouring C. Moreover, since all neighbours of x in H are of degree at most 5 we have |C(z) C(x) C(WH'(x,z))| ≤ 32 < 2Δ(H). According to Lemma 3.1, it follows that H has an acyclic k-colouring, a contradiction.

CONFIGURATION (B7)

If H has a vertex x, having at least dH(x)-5 neighbours of degree at most 5, one of which say z, is of degree 3, then let H' = H-xz. From the fact that H is a minimal counterexample, H' has an acyclic k-colouring C. Let C1 = {C(xy): y ∈ NH(x) and dH(y) > 5}, c2 = {C(xy): y ∈ NH(x), y ≠ z and dH(y) ≤ z}.

If $C(z) \cap C_1 = \emptyset$, then |C(z) C1 C2 (WH'(x,z))| ≤ ΔH + 7. From Lemma 3.1 it follows that H has an acyclic k-colouring, a contradiction.

If $C(z) \cap C_1 \neq \emptyset$, then let z_1, z_2 be neighbours of z in H'. If $C(zz_1) \in C_1$, then there is a colour $C_1\ C(z_1)\ C(z_2)\ C(z)$, with which we can recolour the edge zz_1 with α, to obtain an acyclic k-colouring C' of H'. It follows from the fact that $|C_1\ C_f(z_1)\ C_f(z_2)\ C(z)| \leq 2\Delta H + 4$. Further, if $C(zz_2) \in C_1$, then there is a colour $C_1\ C(z_1)\ C(z_2)\ C(z)\ \{\alpha\}$ which can be used to recolour the edge zz_2 to obtain an acyclic k-colouring C' of H'. It follows from the fact that $|C_1\ C(z_1)\ C(z_2)\ C(z)| \leq 2\Delta(H) + 4$. Further if $C(zz_2) \in C_1$, then there is a colour $C(z_1)C(z_2)\ C(z)\ \{\alpha\}$, which can be used to recolour the edge zz_2 to obtain an acyclic k-colouring C^* of H'. In each situation we are back in the previous case.

THEOREM 3.3

If G is planar, then $\chi'a(G) \leq 2\Delta(G) + 29$.

Proof

It is a known fact that for any planar graph G, $|E(G)| \leq 3|V(G)| - 6$. Note that this property also holds for any subgraph. Therefore we can apply Lemma 3.4 to get the required result.

THEOREM 3.4

If G is 3-fold, then $\chi'a(G) \leq 2\Delta(G) + 29$.

Proof

If G is 3-fold, then G is the union of three forests and thus $|E(G)| \leq 3|V(G)| - 6$. Obviously, this property also holds for any subgraph. Therefore, by Lemma 3.4, $\chi'a(G) \leq 2\Delta(G)+29$.

CONCLUSION

Here we have made use of the discharging method to get some local structure for graphs with bounded number of edges like planar graphs and 2-fold graphs. This was further used to improve the bounds for the acyclic chromatic index. It might be interesting to look at other classes of graphs where the number of edges is linear. It is also interesting to see if the gap between lower and upper bound for planar graphs can be reduced.

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