On 𝕄-Rough Topology

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***Abstract* — The aim of this paper is to extend the concept of topological spaces in rough multisets. The 𝕄-rough topological space is introduced using the concept of Pawlak rough set theory. The basic concepts such as open rough multisets, rough limit points and rough multicontinuity are introduced.**

Keywords— **multisets**, multiset relation, 𝕄-rough topology and 𝕄-rough continuity.

I. INTRODUCTION

In classical set theory, a set is a well defined collection of distinct objects. If repeated occurrences of any object is allowed in a set then a mathematical structure. That is, known as multiset. In 1986, Yagar[8] introduced the concept of multisets and formalized it by defining it as a collection of elements, each considered with certain multiplicity. The same theory was developed by Wayne [1] in 1989. K.P. Girish et al.[5] used the multisets to provide the multiset topology. Topologies on multisets can be associated to IC-bags introduced by Chakrabarthy[2]. A multiset topology is defined a set of multisets as points. Rough set theory was introduced by Pawlak[7] in 1982. A rough set is a tool used in representing, reasoning and decision making. Rough sets theory deals with approximations of msets in terms of the equivalence relations defined on an universe. Rough topology was introduced in 2012 by Mathew et al.[6] in terms of approximations and boundary region of a subset of an universe. The notion of rough multiset was introduced and the properties of it were investigated in [3] and [4]. This paper includes the 𝕄-rough topology which is is an extension of the rough topology in an universal multiset.

II. PRELIMINARIES

1) Definition: [5] An mset M drawn from the set X is represented by a function count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers. Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We present the mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ defined as $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, ..., n$ in the mset M. However those elements which are not included in the mset M have zero count.

2) Definition :[5] A submset R of $M \times M$ is said be an mset relation on M if every member $(m/x, n/y)$ of R has a count product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/x\ R\ n/y$.

3) Definition: [4] Let R be an mset relation on M. The post-mset of $x \in^m M$ is defined as $(m/x)R = \{n/y :$ there exists some k with $(k/x)\ R\ (n/y)\}$ and the pre-mset of $x \in^r M$ is defined as $R(r/x) = \{p/y :$ there exists some q with $(p/y)\ R\ (q/x)\}$.

4) Definition: [4] Let R be any binary mset relation on M in $[X]^m$. Then the mset $\langle n/y \rangle R$ is defined as the intersection of all post-msets containing y with nonzero multiplicity. That is, $\langle n/y \rangle R = \cap\{(m/x)R : y \in^n (m/x)R\}$. Also $R\langle n/y \rangle$ is the intersection of all pre-msets containing y with nonzero multiplicity. That is, $R\langle n/y \rangle = \cap\{R(m/x) : y \in^n R(m/x)\}$.

5) Definition: [4] Let R be a binary mset relation on M. For $N \subseteq M$, a pair of lower and upper mset approximations, $R_L(N)$ and $R_U(N)$ are defined as $R_L(N) = \{m/x : \langle m/x \rangle R \subseteq N\}$, $R_U(N) = \{m/x : \langle m/x \rangle R \cap N \neq \phi\}$ respectively. The pair $(R_L(N), R_u(N))$ is referred to as the rough multiset of N.

III. $\mathbb{M}$-ROUGH TOPOLOGY

1) Definition: Let M be the universe of m-objects and R be an equivalence mset relation on M. For $N \subseteq M$, we define $\tau_R = \{M, \phi, R_L(N), R_U(N), R_B(N)\}$ where $R_L(N)$, $R_U(N)$ and $R_B(N)$ are respectively the lower approximation, the upper approximation and the boundary region of N with respect to R. Note that M and $\phi \in \tau_R$. Since $R_L(N) \subseteq R_U(N)$, $R_L(N) \cup R_U(N) = R_U(N) \in \tau_R$. Also we have $R_U(N) \cup R_B(N) = R_U(N) \in \tau_R$ and $R_L(N) \cup R_B(N) = R_U(N) \in \tau_R$. Also $R_L(N) \cap R_U(N) = R_L(N) \cap R_U(N) = R_L(N) \in \tau_R$ and $R_U(N) \cap R_B(N) = R_B(N) \in \tau_R$ and $R_L(N) \cap R_B(N) = \phi \in \tau_R$.

2) Definition: Let M be the universe of m-objects, R be an equivalence mset relation on M and $\tau_R = \{M, \phi, R_L(N), R_U(N), R_B(N)\}$ where $N \subseteq M$. τ_R satisfies the following axioms.

(i) M and $\phi \in \tau_R$.

(ii) The mset union of the elements of any sub collection of τ_R is in τ_R.

(iii) The mset intersection of the elements of any finite sub collection of τ_R is in τ_R.

τ_R forms a M -topology on M called as the $\mathbb{M}$-rough topology on M with respect N. We call (M, τ_R, N) as the $\mathbb{M}$-rough topological space. Throughout this study, M represents an universe mset and $N \subseteq M$.

3) Example: Let $M = \{3/x, 2/y, 4/z, 7/k\}$ be a any mset. Then $R = \{(3/x, 3/x)/9\ (2/y, 2/y)/4, (4/z, 4/z)/16, (7/k, 7/k)/49, (3/x, 4/z)/12, (4/z, 3/x)/12, (3/x, 7/k)/21, (7/k, 3/x)/21, (2/y, 7/k)/14, (7/k, 2/y)/14\}$ is a equivalence mset relation.

Hence $(3/x)R = \{3/x, 4/z, 7/k\}$, $(2/y)R = \{2/y, 7/k\}$, $(4/z)R = \{4/z, 3/x\}$, $(7/k)R = \{7/k, 3/x, 2/y\}$. Hence the equivalence classes are $\langle 3/x \rangle R = \{3/x\}$, $\langle 2/y \rangle R = \{2/y, 7/k\}$, $\langle 4/z \rangle R = \{4/z, 3/x\}$, $\langle 7/k \rangle R = \{7/k\}$. Let $N = \{3/x, 2/y, 7/k\}$ be a subset of M.

Then $R_L(N) = \{3/x, 2/y, 7/k\}$, $R_U(N) = \{3/x, 2/y, 4/z, 7/k\}$ and $R_B(N) = \{4/z\}$. Therefore, the $\mathbb{M}$-rough topology is $\tau_R = \{\phi, \{3/x, 2/y, 7/k\}, \{4/z\}, M\}$.

4) Definition : If (M, τ_R, N) is an $\mathbb{M}$-rough topological space with respect to N where $N \subseteq M$ and if $A \subseteq M$ then the $\mathbb{M}$-rough interior of A is defined as the union of all $\mathbb{M}$-rough open submsets of A and it is denoted by $\mathrm{M}RInt(A)$.

5) Definition : The $\mathbb{M}$-rough closure of A is defined as the mset intersection of all $\mathbb{M}$-rough closed msets containing A and it is denoted by $MRCl(A)$.

6) Remark : $MRInt(A)$ is the largest $\mathbb{M}$-rough open mset contained in A and $MRCl(A)$ is the smallest $\mathbb{M}$-rough closed mset containing A.

7) Theorem: Let (M,τ_R,N) be the $\mathbb{M}$-rough topological space with respect to N, $N\subseteq M$ and $A,B\subseteq M$. Then

(i) $A\subseteq MRCl(A)$.

(ii) A is $\mathbb{M}$-rough closed iff $A=MRCl(A)$.

(iii) $MRCl(\phi)=\phi$, $MRCl(M)=M$.

(iv) If $A\subseteq B$ then $MRCl(A)\subseteq MRCl(B)$.

(v) $MRCl(A\cup B)=MRCl(A)\cup MRCl(B)$.

(vi) $MRCl(A\cap B)\subseteq MRCl(A)\cap MRCl(B)$.

(vii) $MRCl(MRCl(A))=MRCl(A)$.

8) Theorem: The $\mathbb{M}$-rough closure in an $\mathbb{M}$-rough topological space is the Kuratowski closure operator.

9) Definition: Let m/x be a point in an $\mathbb{M}$-rough topological space M. An mset V in M is said to be a $\mathbb{M}$-rough neighborhood of m/x if $x\in^m G\subset V$ for some $\mathbb{M}$-rough open mset G in M. The collection $MRN_{m/x}$ of all $\mathbb{M}$-rough neighborhoods of m/x is called the $\mathbb{M}$-rough neighborhood system of m/x.

10) Theorem: Let A be a submset of an $\mathbb{M}$-rough topological space M. Then A is $\mathbb{M}$-rough open if and only if each point of A has a $\mathbb{M}$-rough neighborhood contained in A. Consequently, A is $\mathbb{M}$-rough open iff A is a $\mathbb{M}$-rough neighborhood of each of its points.

Proof

Suppose A is $\mathbb{M}$-rough open. Then for each $m/x\in A$, there is an $\mathbb{M}$-rough neighborhood of m/x, namely A itself, contained in A.

Conversely, suppose that each point of A has an $\mathbb{M}$-rough neighborhood contained in A. Let $x\in^m A$. Then by hypothesis, m/x has a $\mathbb{M}$-rough neighborhood $V_{m/x}\subset A$. Hence $x\in^m G_{m/x}\subset V_{m/x}$ for some $\mathbb{M}$-rough open mset $G_{m/x}$. Thus for each $x\in^m A$, $x\in^m G_{m/x}\subset A$ and so $\bigcup_{m/x} G_{m/x}\subset A$. Also $A\subset\bigcup_{m/x} G_{m/x}$. Hence $\mathbb{M}$-rough open $A=\bigcup_{m/x} G_{m/x}$. But each $G_{m/x}$ is $\mathbb{M}$-rough open and so the union, namely A, is also $\mathbb{M}$-rough open.

11) Lemma: Let $MRN_{m/x}$ be the $\mathbb{M}$-rough neighborhood system of a point m/x in the space N. Then $M\subset N$ and

$M\in MRN_{m/x}\Rightarrow N\in MRN_{m/x}$.

$M,N\in MRN_{m/x}\Rightarrow M\cap N\in MRN_{m/x}$.

12) Definition: Let A be a submset of M. Then a point m/x of M is called a $\mathbb{M}$-rough accumulation point (or $\mathbb{M}$-rough limit point) of A if every $\mathbb{M}$-rough neighborhood of m/x intersects A in some point other than m/x itself. The mset of all $\mathbb{M}$-rough accumulation points of A is known as the $\mathbb{M}$-rough derived mset of A and it is denoted by $MRD(A)$.

13) Proposition: The submset $A\subset M$ is $\mathbb{M}$-rough closed if and only if $MRD(A)\subset A$.

14) Lemma: For every submset A of M, $A \cup \mathbb{M}RD(A)$ is a $\mathbb{M}$-rough closed submset of M.

Proof

We show that $M - [A \cup \mathbb{M}RD(A)]$ is $\mathbb{M}$-rough open. Let $x \in^m M - [A \cup \mathbb{M}RD(A)]$. Then by De-Morgan's law, $x \in^m (M - A) \cap (M - \mathbb{M}RD(A))$. Hence $x \notin^m A$ and $x \notin^m \mathbb{M}RD(A)$. Therefore, there exists a $\mathbb{M}$-rough neighborhood V of m/x such that $V \cap A = \phi$. Thus $x \in^m V \subset M - A$. Since V is $\mathbb{M}$-rough open, it is an $\mathbb{M}$-rough neighborhood of each of its points by Theorem 3.11. Hence by no point of V is an $\mathbb{M}$-rough accumulation point of A; so $x \in^m V \subset M - \mathbb{M}RD(A)$. Thus we get $x \in^m V \subset (M - A) \cap (M - \mathbb{M}RD(A))$ and hence we get $x \in^m V \subset M - [A \cup \mathbb{M}RD(A)]$. It follows from Theorem 3.11, $M - [A \cup \mathbb{M}RD(A)]$ is $\mathbb{M}$-rough open and hence $A \cup \mathbb{M}RD(A)$ is an $\mathbb{M}$-rough closed submset of M.

15) Theorem: Let A be a submset of M. Then the following statements hold.

(i) $\mathbb{M}RInt(A)$ is the largest $\mathbb{M}$-rough open submset of A.

(ii) A is $\mathbb{M}$-rough open if and only if $\mathbb{M}RInt(A) = A$.

IV. $\mathbb{M}$-ROUGH CONTINUITY

*1) Definition :*Let (M, τ_R, X) and $(N, \tau_{R'}, Y)$ be two $\mathbb{M}$-rough topological spaces. Then a mapping $f : (M, \tau_R, X) \to (N, \tau_{R'}, Y)$ is $\mathbb{M}$-rough continuous on M if the inverse image of every $\mathbb{M}$-rough open mset in N is $\mathbb{M}$-rough open in M.

2) Example: Let $M = \{4/a, 2/b, 5/c, 7/d, 8/e\}$ and $R = \{(4/a, 4/a)/16, (2/b, 2/b)/4, (5/c, 5/c)/25, (7/d, 7/d)/49, (8/e, 8/e)/64, (4/a, 2/b)/8, (2/b, 4/a)/8, (2/b, 5/c)/10, (5/c, 2/b)/10, (5/c, 7/d)/35, (7/d, 5/c)/35, (8/e, 4/a)/32, (4/a, 8/e)/32\}$ be an equivalence relation.

Hence the equivalence classes are $\langle 4/a \rangle R = \{4/a\}, \langle 2/b \rangle R = \{2/b\}, \langle 5/c \rangle R = \{5/c\}, \langle 8/e \rangle R = \{4/a, 8/e\}$, and $\langle 7/d \rangle R = \{7/d, 5/c\}$. Let $X = \{4/a, 2/b, 5/c\}$ and then $R_L(X) = \{4/a, 2/b, 5/c\}, R_U(X) = \{4/a, 2/b, 5/c, 7/d, 8/e\}$ and $R_B(X) = \{7/d, 8/e\}$. The $\mathbb{M}$-Rough topology is $\tau_R = \{\phi, \{4/a, 2/b, 5/c\}, \{4/a, 2/b, 5/c, 7/d, 8/e\}, \{7/d, 8/e\}, M\}$. Let $N = \{3/x, 4/y, 5/z, 5/k, 6/l\}$ and $R = \{(3/x, 3/x)/9, (4/y, 4/y)/16, (5/z, 5/z)/25, (5/k, 5/k)/25, (6/l, 6/l)/36, (5/z, 5/k)/25, (5/k, 5/z)/25, (3/x, 4/y)/12, (4/y, 3/x)/12, (5/k, 6/l)/30, (6/l, 5/k)/30\}$ be a equivalence mset relation.

The equivalence classes are $\langle 3/x \rangle R = \{3/x, 4/y\}$, and $\langle 4/y \rangle R = \{3/x, 4/y\}$, $\langle 5/z \rangle R = \{5/z, 5/k, 6/l\}$, $\langle 6/l \rangle R = \{6/l, 5/k\}$, $\langle 5/k \rangle R = \{5/k, 6/l\}$. Let $Y = \{4/y, 5/k, 6/l\}$ be a submset of M. Then $R_L(Y) = \{5/k, 6/l\}$, $R_U(Y) = \{3/x, 4/y, 5/k, 5/z, 6/l\}$ and $R_B(Y) = \{3/x, 4/y, 5/z\}$. Then $\mathbb{M}$-Rough topology is $\tau'_R(Y) = \{\phi, \{5/k, 6/l\}, \{3/x, 4/y, 5/z\}, M\}$.

Define $f : M \to N$ as $f(4/a) = 3/x$, $f(2/b) = 4/y$, $f(5/c) = 5/z$, $f(7/d) = 5/k$ and $f(8/e) = 6/l$. Then inverse image is $f^{-1}(\{\phi\}) = \phi$, $f^{-1}(\{5/k, 6/l\}) = \{7/d, 8/e\}$ and $f^{-1}(\{3/x, 4/y, 5/z\}) = \{4/a, 2/b, 5/c\}$, $f^{-1}(\{N\}) = M$. That is, the inverse of every $\mathbb{M}$-rough open mset in N is $\mathbb{M}$-rough open mset in M.

3) Theorem : A function $f : (M, \tau_R, X) \to (N, \tau_{R'}, Y)$ is $\mathbb{M}$-rough continuous if and only if the inverse image of every $\mathbb{M}$-rough closed mset in N is $\mathbb{M}$-rough closed in M.

Proof

Let f be $\mathbb{M}$-rough continuous and F be $\mathbb{M}$-rough closed in N. That is, $N-F$ is $\mathbb{M}$-rough open in N. Since f is $\mathbb{M}$-rough continuous, $f^{-1}(N-F)$ is $\mathbb{M}$-rough open in M. That is, $M-f^{-1}(F)$ is $\mathbb{M}$-rough open in M. Therefore, $f^{-1}(F)$ is $\mathbb{M}$-rough closed in M. Thus the inverse image of every $\mathbb{M}$-rough closed mset in N is $\mathbb{M}$-rough closed in M, if f is $\mathbb{M}$-rough continuous on M.

Conversely, let the inverse image of every $\mathbb{M}$-rough closed mset in N be $\mathbb{M}$ -rough closed in M. Let G be $\mathbb{M}$-rough open in N. Then $N-G$ is $\mathbb{M}$-rough closed mset in N. Then $f^{-1}(N-G)$ is $\mathbb{M}$-rough closed in M. That is, $M-f^{-1}(G)$ is $\mathbb{M}$-rough closed in M. Therefore, $f^{-1}(G)$ is $\mathbb{M}$-rough open in M. That is, f is $\mathbb{M}$-rough continuous on M.

4) Theorem: A function $f:(M,\tau_R,X)\to(N,\tau_{R'},Y)$ is $\mathbb{M}$-rough continuous if and only if $f(\mathbb{M}RCl(A))\subseteq \mathbb{M}RCl(f(A))$ for every submset A of M.

5) Remark: If $f:(M,\tau_R,X)\to(N,\tau_{R'},Y)$ is $\mathbb{M}$-rough continuous then $f(\mathbb{M}RCl(A))$ is not necessarily equal to $\mathbb{M}RCl(f(A))$ for every $A\subseteq M$.

6) Theorem: Let (M,τ_R,X) and $(N,\tau_{R'},Y)$ be two $\mathbb{M}$-rough topological spaces where $X\subseteq M$ and $Y\subseteq N$. Then $\tau_{R'}(Y)=\{N,\phi,R_{L'}(Y),R_{U'}(Y),R_{B'}(Y)\}$ and its basis is given by $\mathrm{B}_{R'}=\{N,R_{L'}(Y),R_{B'}(Y)\}$. A function $f:(M,\tau_R,X)\to(N,\tau_{R'},Y)$ is $\mathbb{M}$-rough continuous if and only if the inverse image of every member of $\mathrm{B}_{R'}$ is $\mathbb{M}$-rough open in M.

Proof

Let f be $\mathbb{M}$-rough continuous on M. Let $A\in \mathrm{B}_{R'}$. Then A is $\mathbb{M}$-rough open in N. That is, $A\in\tau_{R'}(Y)$. Since f is $\mathbb{M}$-rough continuous, $f^{-1}(A)\in\tau_R(X)$. That is, inverse image of every member of $\mathrm{B}_{R'}$ is $\mathbb{M}$-rough open in M. Conversely, let the inverse image of every member of $\mathrm{B}_{R'}$ be $\mathbb{M}$-rough open in M. Let G be $\mathbb{M}$-rough open in N. Then $G=\bigcup\{B:B\in\mathrm{B}_1\}$, where $\mathrm{B}_1\subset\mathrm{B}_{R'}$. Then $f^{-1}(G)=f^{-1}(\bigcup\{B:B\in\mathrm{B}_1\})=\bigcup\{f^{-1}(B):B\in\mathrm{B}_1\}$ where each $f^{-1}(B)$ is $\mathbb{M}$-rough open in M and hence their union which is $f^{-1}(G)$ is $\mathbb{M}$-rough open in M. Thus f is $\mathbb{M}$-rough continuous on M.

7) Theorem: A function $f:(M,\tau_R,X)\to(N,\tau_{R'},Y)$ is $\mathbb{M}$-rough continuous if and only if for every submset B of N, $\mathbb{M}RCl(f^{-1}(B))\subseteq f^{-1}(\mathbb{M}RCl(B))$.

8) Theorem: A function $f:(M,\tau_R,X)\to(N,\tau_{R'},Y)$ is $\mathbb{M}$-rough continuous if and only if for every submset B of N, $f^{-1}(\mathbb{M}RInt(B))\subseteq \mathbb{M}RInt(f^{-1}(B))$.

9) Theorem : If (M,τ_R,X) and $(N,\tau_{R'},Y)$ are $\mathbb{M}$-rough topological spaces with respect to $X\subseteq M$ and $Y\subseteq N$ respectively then for any function $f:M\to N$ the following are equivalent

(i) f is $\mathbb{M}$-rough continuous.

(ii) The inverse image of every -rough closed mset in N is $\mathbb{M}$-rough closed in M.

(iii) $f(\mathbb{M}RCl(A))\subseteq \mathbb{M}RCl(f(A))$ for every submset A of M.

(iv) The inverse image of every member of the $\mathbb{M}$-basis $B_{R'}$ of $\tau_{R'}(Y)$ is $\mathbb{M}$-rough open in M.

(v) $\mathrm{M}RCl(f^{-1}(B)) \subseteq f^{-1}(\mathrm{M}RCl(B))$ for every submset B of N.

(vi) $f^{-1}(\mathrm{M}RInt(B)) \subseteq \mathrm{M}RInt(f^{-1}(B))$ for every submset B of N.

REFERENCES

[1] Blizard Wayne. D, On multiset theory, Notre Dame Journal of Formal Logic, Vol. 30, No. 1 (1989), pp. 36-66.

[2] Chakrabarty. K, Bags with interval counts, Foundations of Computing and Decision Sciences, Vol. 25, No. 1,(2000), pp. 23-36.

[3] El-sayed. A and Abo-Tabl, On topological structure of generalized rough multisets, Life Science Journal, Vol. 11, No. 4 (2014), pp. 290-299.

[4] El-sheik S.A and Zakaria. A, On Note on rough multiset and its multiset topology, Annals of Fuzzy Mathematics and informatics, Vol. X, No. X, (2015) pp. 1-XX.

[5] [5]. Girish K.P and Sunil Jacob John, On rough multisets and information multisystems, Advances in Decision Sciences, Vol. 2011, (2011), pp. 17.

[6] Pawlak .Z, Rough sets, International Journal of Computer Science, Vol. 11, No. 5, (1982), pp. 341-356.

[7] Mathew. Boby P and Sunil Jacob John, On Rough Topological Spaces, International Journal of Mathematical Archive, Vol. 3, No. 9, (2012), pp.3413-3421.

[8] Yager R.R, On the theory of bags, Int. J. Gen. Syst., Vol. 13, (1986), pp. 23-37.