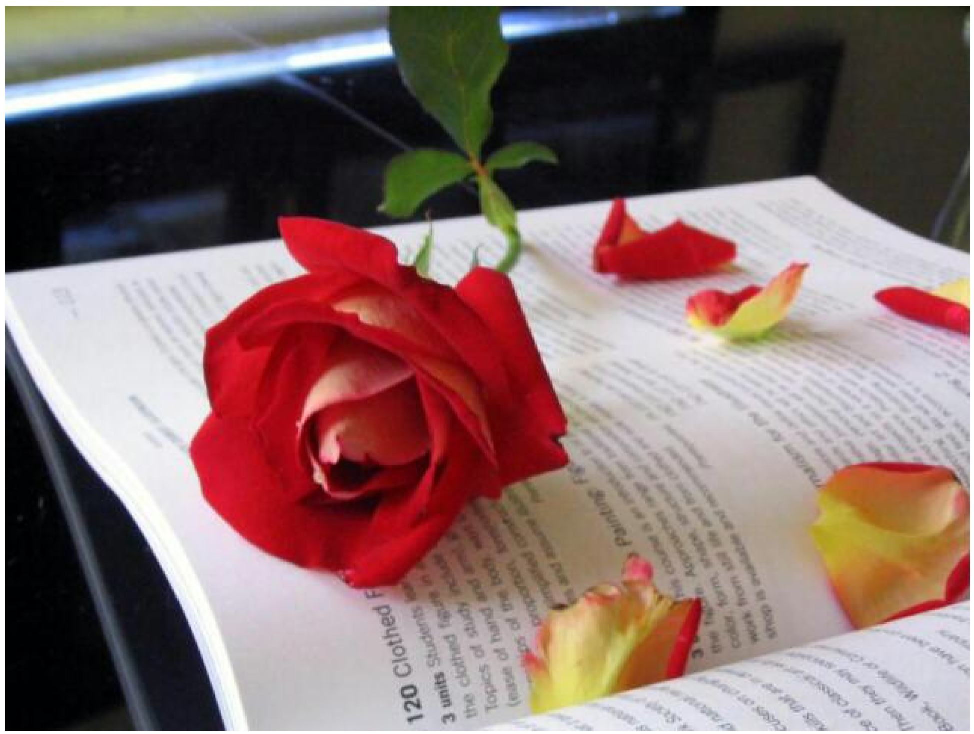
Finite Abelian groups

By Dr.Mrs F.M.Mary Latha,
Associate Professor of Maths



Theorem

Every finite abelian group is a direct product of cyclic groups

Proof

We know that Every finite abelian group is the direct of its Sylow subgroup.

∴ If we prove that every such sylow subgroup is the direct product of cyclic subgroups. We could the result together and realise G is the direct product of cyclic groups.

Let G be an abelian group of order p^n.

Let us find elements $a_1, a_2, \dots, a_k$ in G such that every element x in G can be uniquely written as

$$x = a_1^{\alpha_1} a_2^{\alpha_2} \ldots\ldots a_k^{\alpha_k}$$

i.e. G is a direct product of cyclic groups generated by $a_1\ a_2\ \ldots\ a_k$ respectively.

Let a_1 be the element of G of maximum order say p^{n_1}

i.e. $a_1^{p^{n_1}} = e$

Let $A_1 = < a_1 >$ be the cyclic group generated by a_1. It is a normal subgroup of G.

$\overline{G} = \frac{G}{A_1}$ is the quotient group.

Pick an element b_2 in G such that $\overline{b_2} = b_2 A_1$ the image of b_2 in $\overline{G}$ has the maximal order p^{n_2}.

Since the order of $\overline{b_2}$ divides the order b_2 and a_1 is of maximal order.

We have $p^{n_1} > p^{n_2} \Rightarrow n_1 \geq n_2$ ⇢ (1)

Step 1

We claim $A_1 \cap < b_2 > = \{e\}$

Suppose $A_1 \cap < b_2 > \neq \{e\}$

Then there exists an element belongs to both A_1 and $< b_2 >$

i.e. ${b_2}^{p^{n_2}} \in A_1$ but $A_1 = < a_1 >$

${b_2}^{p^{n_2}} = {a_1}^i$ for some positive integer i

Consider $({a_1}^i)^{p^{n_1-n_2}} = ({b_2}^{p^{n_2}})^{p^{n_1-n_2}}$

$= {b_2}^{p^{n_2-n_1-n_2}}$

$$= b_2^{p^{n_1}}$$

$$= e$$

$(a_1^{\,i})^{p^{n_1-n_2}} = e$ but a_1 is of order p^{n_1}

i.e. p^{n_1} is the least positive integer such that

$a_1^{p^{n_1}} = e$

$a_1^{ip^{n_1-n_2}} \Rightarrow p^{n_1} / ip^{n_1-n_2}$

i.e. $ip^{n_1-n_2} = k\, p^{n_1}$

$i = k\, p^{n_2}$

i.e. $p^{n_2} / i \Rightarrow i = j \;\; p^{n_2}$ for some positive integer j

$a_1^{\,i} = a_1^{\,jp^{n_2}} = b_2^{p^{n_2}}$

choose $a_2 = a_1^{-j}\, b_2$

then $a_2^{p^{n_2}} = (a_1^{-j}\, b_2)^{p^{n_2}}$

$$= a_1^{-jp^{n_2}} \, . \, b_2^{p^{n_2}}$$

$a_2^{p^{n_2}} = a_1^{0} = e$ s.t $n_2 \leq n_1$

we take $A_2 = \langle a_2 \rangle$

Step 2

we can prove that $A_1 \cap A_2 = (e)$.

Suppose there exist an element $a_2^{t} \in A_1$ but

$a_2 = a_1^{-j}\, b_2$

$\therefore (a_1^{-j}\, b_2)^{t} \in A_1$

$a_1^{-jt}\, b_2^{t} \in A_1 \Rightarrow b_2^{t} \in A_1$

But p^{n_2} is the first power of b_2 such that $b_2^{p^{n_2}} \in A_1$

$$\therefore p^{n_2} / t$$

$$t = k\,p^{n_2}$$

$$a_2{}^{t} = a_2{}^{kp^{n_2}}$$

$$= (a_2{}^{p^{n_2}})^{k}$$

$$a_2{}^{t} = e^{k} = e$$

So if an element $a_2{}^{t} \in A_1 \cap A_2$

$$a_2{}^{t} = (e)$$

Hence $A_1 \cap A_2 = (e)$

Again we pickup $b_3 \in G$ such that $\overline{b_3}$ is the image of b_3 in $\frac{G}{A_1A_2}$ is of maximal order say p^{n_3}

then $n_3 \leq n_2 \leq n_1$ because p^{n_2} is the maximal order of any element in $\overline{G}$ which belongs to A_1

$b_3^{p^{n_2}} \in A_1$

$b_3^{p^{n_2}} \in A_1 A_2$ but $b_3^{p^{n_3}} \in A_1 A_2$

$p^{n_2} \geq p^{n_3} \Rightarrow n_3 \leq n_2 \leq n_1$

Then $b_3^{p^{n_3}} \in A_1 A_2 \Rightarrow b_3^{p^{n_3}} = a_1^{i_1} a_2^{i_2}$ for some intege i_1 and i_2

Step 3

we can prove that p^{n_3}/i_1 and p^{n_3}/i_2

Consider $(a_1^{i_1} a_2^{i_2})^{p^{n_2-n_3}} = (b_3^{p^{n_3}})^{p^{n_2-n_3}}$

$= b_3^{p^{n_3-n_2-n_3}}$

$= b_3^{p^{n_2}} \in A_1$

i.e. $(a_1^{i_1} a_2^{i_2})^{p^{n_2-n_3}} \in A_1$.

Since $A_1 = < a_1 >$ but $(a_1{}^{i_1})^{p^{n_2-n_3}}$

and so $(a_1{}^{i_1}\ a_2{}^{i_2})^{p^{n_2-n_3}} \in A_1$ and $(a_1{}^{i_1})^{p^{n_2-n_3}} \in A_1$

$$\Rightarrow (a_2{}^{i_1})^{p^{n_2-n_3}} \in A_1$$

$$a_2{}^{i_1 p^{n_2-n_3}} \in A_1 \text{ but } a_2{}^{p^{n_2}} = e$$

$$\Rightarrow {}^{p^{n_2}}/_{i_2.p^{n_2-n_3}}.$$

$$i_2.p^{n_2-n_3} = k\,p^{n_2}$$

$$i_2.p^{-n_3} = k$$

$$i_2 = kp^{n_3} \Rightarrow {}^{p^{n_3}}/_{i_2}$$

Illy using $b_3{}^{p^{n_1}} = e$

Consider $(a_1{}^{i_1}\ a_2{}^{i_2})^{p^{n_1-n_3}} = (b_3{}^{p^{n_3}})^{p^{n_1-n_3}}$

Consider $(a_1^{i_1}\ a_2^{i_2})^{p^{n_1-n_3}} = (b_3^{p^{n_3}})^{p^{n_1-n_3}}$

$= b_3^{p^{n_3-n_1-n_3}}$

$= b_3^{p^{n_1}}$

$(a_1^{i_1})^{p^{n_1-n_3}}\ (a_2^{i_2})^{p^{n_1-n_3}} = e$

$(a_1^{i_1})^{p^{n_1-n_3}} = (a_2^{-i_2})^{p^{n_1-n_3}} \in A_1 \cap A_2 = \{e\}$

$(a_1^{i_1})^{p^{n_1-n_3}} = e$

$(a_1^{i_1})^{p^{n_1}}\ (a_1^{i_1})^{p^{-n_3}} = e$

$(a_1^{p^{n_1}})^{i_1}\ (a_1^{i_1})^{p^{-n_3}} = e$

$(a_1^{i_1})^{p^{-n_3}} = e$

$a_1^{i_1} = a_1^{p^{n_3}} = e \Rightarrow p^{n_3}/i_1$

Hence we get p^{n_3}/i_1 and p^{n_3}/i_2

Let $i_1 = j_1 p^{n_3}$ and $i_2 = j_2 p^{n_3}$

Write an element $a_3 = a_1{}^{-j_1} a_2{}^{-j_2} b_3$

Consider the cyclic group $A_3 = < a_3 >$

$$a_2{}^{p^{n_3}} = (a_1{}^{-j_1} a_2{}^{-j_2} b_3)^{p^{n_3}}$$

$$= a_1{}^{-j_1 p^{n_3}} a_2{}^{-j_2 p^{n_3}} b_3{}^{p^{n_3}}$$

$$= (a_1{}^{-i_1} . a_1{}^{-i_2}) b_3{}^{p^{n_3}}$$

$$= e$$

$O(a_3) = p^{n_3}$

$p^{n_3} < p^{n_2} < p^{n_1}$

Hence $n_3 \leq n_2 \leq n_1$

Step 4

Next we claim $A_3 \cap (A_1A_2) = (e)$

For if $h \in A_3$ and $h \in A_1A_2$

$h \in A_3 = < a_3 >$

$\therefore h = {a_3}^t$

${a_3}^t \in A_1A_2$

i.e. $({a_1}^{-j_1}\ {a_2}^{-j_2}\ b_3)^t \in A_1A_2$

But $({a_1}^{-j_1}\ {a_2}^{-j_2})^t \in A_1A_2$

So ${b_3}^t \in A_1A_2$

Since ${b_3}^{p^{n_3}} = e$

p^{n_3} / t

$t = k\, p^{n_3}$

$h = a_3{}^t = a_3{}^{kp^{n_3}}$

But $a_3{}^{p^{n_3}} = e$

$h = (a_3{}^{p^{n_3}})^k$

$= e^k$

$h = e$

$\therefore A_3 \cap (A_1 \cap A_2) = e$

Continuing this way we get a cyclic groups

$A_1 = < a_1 >$ of order p^{n_1}

$A_2 = < a_2 >$ of order p^{n_2}

$A_3 = < a_3 >$ of order p^{n_3}

………………

$A_k = < a_k >$ of order p^{n_k}

with $p^{n_k} \leq p^{n_{k-1}} \leq \ldots\ldots \leq p^{n_3} \leq p^{n_2} \leq p^{n_1}$

$\Rightarrow n_k \leq n_{k-1} \leq \cdots \leq n_3 \leq n_2 \leq n_1$

such that G = $A_1 A_2 A_3 \ldots \ldots A_k$

$A_i \cap (A_1 A_2 \ldots A_{i-1} A \ldots A_k) = e \ \forall$ i= 1,2,….,k

Hence every element x∈G has a unique representation as $x = a_1^{\alpha_1} a_2^{\alpha_2} \ldots\ldots a_k^{\alpha_k}$

$a_i^{\alpha_i} \in A_i$ which is a cyclic group of order p^{n_i}

G is a direct product of cyclic groups $A_1 A_2 A_3 \ldots \ldots A_k$


Thanks