
Welcome

ROLES OF THE POISSON AND EXPONENTIAL DISTRIPUTION

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II M.Sc MATHS

Consider the queueing situation in which the number of arrivals and departures during an interval of time is controlled by the following conditions

CONDITION:1

The probability of an event occurring between times t and t+h depends only on the length of h.

CONDITION:2

The probability of an event occurring during a very small time interval h is positive but less than 1.

CONDITION:3

At most one event can occur during a very small time interval h.

We shall find the general expression for $P_n(t)$

$P_n(t)$ =Probability of n events occurring during time t.

Condition 1 implies that $P_n(t)$ has stationary independent increments

For n=0

$p_0(t+h)$ = Probability of no occurrences in time interval t+h

= Probability of no occurrence in time t and no accurrence in time t to t+h.

= Pr{no accurrence in time t}.

Pr{no accurrence in time t to t+h}

$p_0(t+h) = p_0(t)\ .p_0(h)$

By condition 2 we have 0<$p_0(h)$<1 for very small h.

By poisson postulates, for a small interval of time h,

$$
\begin{aligned}
p_0(h) &= 1-\lambda(h)+o(h)\\
p_1(h) &= \lambda(h)+o(h)\\
p_n(h) &= o(h) \quad \text{for all } n \geq 2\\
p_0(t+h) &= p_0(t)\,(1-\alpha(h)+o(h))\\
&= p_0(t) - \alpha h p_0(t) + o(h)
\end{aligned}
$$

$$p_0(t+h) - p_0(t) = -\boldsymbol{\alpha h p_0(t) + o(h)}$$

$$\frac{p_0(t+h) - p_0(t)}{h} = \alpha p_0(t) + o(h)$$

$\lim h \to 0$

$$p_0'(t) = -\boldsymbol{\alpha}\, p_0(t)$$

$$\frac{p_0'(t)}{p_0(t)} = -\boldsymbol{\alpha}$$

integrating

$$\int \frac{p_0'(t)}{p_0(t)} \boldsymbol{dt} = \int \boldsymbol{-\alpha dt + A}$$

t=0,

$$\log p_0(t) = -\alpha t + A$$

$$\log p_0(0) = A$$

$$\log 1 = A$$

$$A = 0$$

Therefore

$$\log p_0(t) = -\alpha t$$

$$p_0(t) = e^{-\alpha t}$$

Where α is a positive constant and α denoting the rate of arrivals (departures) per unit time.

For h>0 and sufficiently small, we have

$$p_0(t+h) = p_0(t) \, . \, p_0(h)$$

$$p_0(h) = e^{-\alpha h}$$

$$= 1 - \alpha h + \frac{(\alpha h)^2}{2!} - \frac{(\alpha h)^3}{3!} + \cdots\cdots$$

$$p_0(h) \cong 1 - \alpha h$$

Condition 3 gives the atmost 1 event can occur in the small interval of time h.

$$\therefore p_0(h) + p_1(h) = 1$$

$$p_1(h) = 1 - p_0(h)$$

$$= 1 - (1 - \alpha h)$$

$$p_1(h) \cong \alpha h$$

This means that the probability of an event occurring during a small interval h is directly proportional to h.

$\therefore p_n(t)$ is completely random.

(ie) The probability of n^{th} occurrence of the event in time t is independent of the time of the $(n-1)^{th}$ occurrence

Let $f(t)$ = Probability density function of the time interval t between the occurrence of successive events ,

$F(t)$ = Cumulative density function (CDF) of t,

$$[\ f(t) = F'(t) \Rightarrow F(t) = \int_0^t f(x)dx]$$

$$F(t) = \int_0^t f(x)dx$$

Pr{t≥T}= $F(t)$

If T is the time interval since the occurrence of the last event,then we have

Pr{interevent time is not less than T}= Pr{no events occur during T}

Pr{t≥T}= $p_0(t)$

Since f(t) is the probability density function of t and

$$p_0(T) = e^{-\alpha T}, T > 0$$

We have

Pr{t≥T}= $e^{-\alpha T}$

$$\int_T^{\infty} f(t)dt = e^{-\alpha T}$$

(Or) using the definition of F(T), we have

$$1 - F(T) = e^{-\alpha T}, T > 0$$

Differentiating both sides with respect to T, we obtain

$$f(T) = \alpha e^{-\alpha T}, T > 0$$

Which is exponential distribution.

The result above yields two conclusions

1.For the process described by the probabilities $p_n(t)$,the time between the occurrence of successive events must follow an exponential distribution.

2.The expected value of the exponential distribution

$$E[T] = \frac{1}{\alpha} \text{ time units}$$

represents the average time interval between successive occurrences of events

Thus $$\frac{1}{E\{T\}} = \alpha \ events/\ unit\ time$$

Must represent the rate (per unit time)at which events are generated .This is the reason we indicated earlier that α represents arrivals (departures).

3.The exponential distribution has the unique property that the time until the next event occurs is independent of the time that elapsed since the occurrence of the last event. This result is equivalent to stating that

$$P\{t > T + S | t > S\} = P\{t > T\}$$

Where t is the random variable describing the interevent time and S is the occurrence time of the last event .

To show that this probability is true for the exponential distribution , Consider

$$P\{t > T + S | t > S\} = \frac{P\{t > T + S, t > S\}}{P\{t > S\}} = \frac{P\{t > T + S\}}{P\{t > S\}}$$

$$= \frac{e^{-\alpha(T+S)}}{e^{-\alpha S}} = e^{-\alpha T}$$

$= P\{t > T\}$

This property is usually referred to as forgetfulness or lack of memory of the exponential distribution.


THANK YOU